

# Linearized stability analysis of nonlinear partial differential equations

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**Abstract**—Lyapunov’s indirect method is an attractive method for analyzing stability of non-linear systems since only the stability of the corresponding linearized system needs to be determined. Unfortunately, the proof for finite-dimensional systems does not generalize to infinite-dimensions. In this paper a unified approach to Lyapunov’s indirect method for infinite-dimensional system is described. It is shown how existing sufficient conditions fit this framework and a new sufficient condition is presented.

## I. INTRODUCTION

Stability theory for finite-dimensional nonlinear systems is well established [1]–[3]. It is natural to generalize this theory for infinite-dimensional systems. Lyapunov’s direct method generalizes to infinite-dimensional systems in a straightforward manner. LaSalle’s invariance principle also generalizes, provided that the orbit of the system is pre-compact. [4]–[7]. However, finding a Lyapunov function for nonlinear infinite-dimensional systems is challenging as is showing the pre-compactness of the orbit of the system. Lyapunov’s indirect method is appealing as it is a systematic approach and the theory for the stability of linear systems is well understood. However, the proof of Lyapunov’s indirect method for finite-dimensional systems does not generalize to infinite-dimension dimensions.

Some results on linearization as a way of analyzing the stability of nonlinear infinite-dimensional systems have been obtained [8]–[10]. For instance, Smoller [10, Theorem 11.17] showed that if the nonlinear operator in the model is locally Lipschitz continuous, continuously Fréchet differentiable and also a condition related to the nonlinear operator being twice continuously differentiable is satisfied, then the  $C_0$ -semigroup generated by the nonlinear system is continuously Fréchet differentiable. Using this result, Smoller [10, Theorem 11.22] also showed that if the linearized system at an equilibrium generates an exponentially stable  $C_0$ -semigroup, then the nonlinear system generates a locally exponentially stable  $C_0$ -semigroup in a neighbourhood of that equilibrium. Kato [9, Corollary 2.2] relaxed the condition of the nonlinear operator and was able to achieve the same stability result. In [11], the Fréchet differentiability of the  $C_0$ -semigroup corresponding to a class of quasilinear systems is proved under different conditions from the ones in [9], [10].

In this paper, it is shown that the Fréchet differentiability of the  $C_0$ -semigroup generated by the nonlinear infinite-dimensional system plays an important role in the justification of Lyapunov’s indirect method. Exponential stability

as opposed to asymptotic stability is also important. The nonlinear Kuramoto-Sivashinsky equation is presented as an example. A new sufficient condition for Fréchet differentiability is also presented.

## II. PRELIMINARIES

Consider the nonlinear time-invariant system defined on a Banach space  $X$  with norm  $\|\cdot\|$

$$\begin{aligned} \dot{z}(t) &= F(z(t)), \quad t \geq 0 \\ z(0) &= z_0, \end{aligned} \quad (1)$$

where  $z_0$  is the initial condition, the nonlinear operator  $F: \mathcal{D}(F) \subset X \rightarrow X$  is densely defined on  $X$ . Assume that this system is well-posed; that is, it has a unique solution that can be written

$$z(t) = S(t)z_0,$$

where  $S(t)$  is a nonlinear  $C_0$ -semigroup on  $X$  generated by the operator  $F$ .

Let  $z_e$  be an equilibrium of (1); that is, with  $F(z_e) = 0$ . The standard definitions of stability for an equilibrium point are used here; see for example, [6]. Many infinite-dimensional dynamical systems possess an infinite number of stable equilibria. A simple example is the heat equation with Neumann boundary conditions.

A set of equilibrium points can also be characterized as stable.

*Definition 2.1:* [7, Definition 2.6] (Stable Equilibrium Set)

Let  $E$  be the set of all equilibria to (1). The set  $E$  is said to be stable if for every  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $dist_X(z_0, E) < \delta$ , then

$$dist_X(z(t), E) < \varepsilon, \quad t \geq 0$$

where  $dist_X(z, E) = \inf\{\|z - y\| : y \in E\}$ .

*Definition 2.2:* [7, Definition 2.6] (Globally Asymptotically Stable Equilibrium Set)

Let  $E$  be the set of all equilibria to (1). The set  $E$  is said to be globally asymptotically stable if it is stable and for every  $z_0 \in X$ ,

$$\lim_{t \rightarrow \infty} dist_X(z(t), E) = 0.$$

## III. LINEARIZED STABILITY OF NONLINEAR SYSTEMS

The following result is well-known.

*Theorem 3.1:* (e.g. [1, Theorem 3.19])

Consider the nonlinear system (1) and assume that  $X$  is

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finite-dimensional. Assume also that  $F$  is differentiable and define

$$A = \left. \frac{\partial F}{\partial z} \right|_{z=z_e}$$

to be the linearization of (1). Then

- (1) if  $Re\lambda < 0$  for all  $\lambda \in \sigma(A)$ , then the equilibrium  $z_e$  to (1) is exponentially stable where  $\sigma(A)$  is the spectrum of  $A$ .
- (2) if there exists  $\lambda \in \sigma(A)$  such that  $Re\lambda > 0$ , then the equilibrium  $z_e$  to (1) is unstable.

The proof of Theorem 3.1, see for example, [1, Thm 3.1, Thm. 3.12], relies on showing that a Lyapunov function for the linear system, which can be easily constructed, is also a Lyapunov function for the non-linear system in a region around the equilibrium point. It follows then that that  $z_e$  is locally exponentially stable. The proof of instability is similar.

Using Lyapunov's indirect method for nonlinear infinite-dimensional systems requires a justification similar to Theorem 3.1 that the stability of the linearized systems reflects the stability of the nonlinear system. However, generalization of the proof to infinite-dimensions is not straightforward since the operator  $F$  is typically not Fréchet differentiable; in fact it is generally an unbounded operator. There are two issues that need to be addressed. First, how to linearize the nonlinear system defined on a Banach space  $X$ ? Second, what conditions guarantee that the stability of the linearized infinite-dimensional system is the same as the nonlinear system? That is, if the linearized system is stable or unstable, then does the same conclusion apply to the original nonlinear system?

**Definition 3.2:** [12, Definition 3.1.1] Consider an operator  $F : X \rightarrow X$  defined on a normed linear space  $X$ . The operator  $F$  is Fréchet differentiable at  $z_0$  if there exists a bounded linear operator  $DF(z_0) : X \rightarrow X$  such that for all  $h \in X$

$$\lim_{h \rightarrow 0} \frac{\|F(z_0 + h) - F(z_0) - DF(z_0)h\|}{\|h\|} = 0, \quad (2)$$

That is,

$$F(z_0 + h) - F(z_0) = DF(z_0)h + \omega(z_0, h),$$

where

$$\lim_{\|h\| \rightarrow 0} \frac{\|\omega(z_0, h)\|}{\|h\|} \rightarrow 0.$$

The operator  $F$  is said to be Fréchet differentiable if it is Fréchet differentiable at every  $z_0 \in X$ .

The next theorem demonstrates that Fréchet differentiability of the  $C_0$ -semigroup generated by the nonlinear infinite-dimensional system plays a key role in the validity of Lyapunov's indirect method. A similar result was shown in [13] under the condition that the number of unstable eigenvalues corresponding to the linearized system is finite. In the next theorem this assumption is not required. For more details, see [14], [15].

**Theorem 3.3:** Consider the nonlinear system (1) defined on a Banach space  $X$ . Assume that the nonlinear operator  $F : \mathcal{D}(F) \subset X \rightarrow X$  generates a nonlinear  $C_0$ -semigroup  $S(t)$ . Let  $z_e$  be an equilibrium for the above system (1) and suppose that  $S(t)$  is Fréchet differentiable at  $z_e$ .

- (i) If  $z_e$  is an exponentially stable equilibrium of the linearized system, then  $z_e$  is a locally exponentially stable equilibrium of the nonlinear system (1).
- (ii) If the linearized system is unstable, then the nonlinear system (1) is locally unstable.

**Proof:** (i) Let  $T_{z_e}(t)$  be the Fréchet derivative of  $S(t)$  at  $z_e$ . It follows that

$$S(t)z_0 - S(t)z_e = T_{z_e}(t)(z_0 - z_e) + \omega(z_e, z_0 - z_e),$$

where

$$\lim_{\|z_0 - z_e\| \rightarrow 0} \frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} = 0. \quad (3)$$

That is, for any  $t > 0$ ,  $\varepsilon_t > 0$ , there exists  $\delta > 0$  such that if  $\|z_0 - z_e\| < \delta$ ,

$$\frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} < \varepsilon_t.$$

Furthermore, since the  $C_0$ -semigroups  $S(t)$  and  $T_{z_e}(t)$  are continuous in  $t$ , then the function  $\omega$  is continuous in  $t$  and for  $M \geq 1$  and  $\gamma > 0$  such that for all  $z_0 \in X$

$$\|T_{z_e}(t)z_0 - z_e\| \leq Me^{-\gamma t}\|z_0 - z_e\|, \quad t \geq 0, \quad (4)$$

then there exists  $\varepsilon > 0$ ,  $\bar{t} < \infty$ , such that for  $\tau \in [0, \bar{t}]$

$$\begin{aligned} \|S(\tau)z_0 - z_e\| &\leq \|T_{z_e}(\tau)(z_0 - z_e)\| + \|\omega(z_e, z_0 - z_e)\| \\ &\leq Me^{-\gamma\tau}\|z_0 - z_e\| + \varepsilon\|z_0 - z_e\| \\ &= C\|z_0 - z_e\| \end{aligned}$$

where  $C = M + \varepsilon$ . Choose  $\bar{t} = \ln(4M)/\gamma > 0$ , then using (4)

$$\|T_{z_e}(\bar{t})z_0 - z_e\| \leq \frac{1}{4}\|z_0 - z_e\|. \quad (5)$$

It follows that

$$\begin{aligned} \lim_{\|z - z_e\| \rightarrow 0} \left\| \frac{S(\bar{t})z_0 - S(\bar{t})z_e - T_{z_e}(\bar{t})z_0 + T_{z_e}(\bar{t})z_e}{z_0 - z_e} \right\| \\ = \lim_{\|z - z_e\| \rightarrow 0} \left\| \frac{S(\bar{t})z_0 - T_{z_e}(\bar{t})z_0}{z_0 - z_e} \right\| = 0 \end{aligned}$$

and hence, there exists  $\delta > 0$  such that if  $\|z_0 - z_e\| < \delta$ , then

$$\|S(\bar{t})z_0 - T_{z_e}(\bar{t})z_0\| \leq \frac{1}{4}\|z_0 - z_e\|. \quad (6)$$

Using (5) and (6),

$$\|S(\bar{t})z_0 - z_e\| \leq \frac{1}{2}\|z_0 - z_e\| = e^{-\ln 2}\|z_0 - z_e\|. \quad (7)$$

Let  $k > 0$  be an integer, then using the semigroup property and (7),

$$\begin{aligned} \|S(k\bar{t})z_0 - z_e\| &= \|S^k(\bar{t})z_0 - z_e\| \\ &= \|S(\bar{t})S^{k-1}(\bar{t})z_0 - z_e\| \\ &\leq e^{-\ln 2}\|S^{k-1}(\bar{t})z_0 - z_e\| \\ &\leq e^{-(\ln 2)k}\|z_0 - z_e\|. \end{aligned} \quad (8)$$

For  $t > 0$ , let  $k = [t/\bar{t}]$  and  $\tau = t - k\bar{t}$ . Then  $\tau \in [0, \bar{t}]$  and using the semigroup property, (5) and (8),

$$\begin{aligned} \|S(t)z_0 - z_e\| &= \|S(k\bar{t} + \tau)z_0 - z_e\| \\ &= \|S(\tau)S(k\bar{t})z_0 - z_e\| \\ &\leq C\|S(k\bar{t})z_0 - z_e\| \\ &\leq Ce^{-(\ln 2)k}\|z_0 - z_e\| \\ &\leq Ce^{-\alpha t}\|z_0 - z_e\| \end{aligned}$$

for  $\alpha \leq \ln 2/\bar{t}$ . This implies that the equilibrium  $z_e$  to the nonlinear system is locally exponentially stable.

(ii) The result is shown by proving the contrapositive. Let  $z_e$  be a locally stable equilibrium to the nonlinear system (1). Since  $T_{z_e}(t)$  is a linear operator and  $z_e$  is an equilibrium,

$$S(t)z_0 - z_e = T_{z_e}(t)z_0 - z_e + \omega(z_e, z_0 - z_e). \quad (9)$$

The definition of locally stable equilibrium of the nonlinear system implies that for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that if  $\|z_0 - z_e\| < \delta$ , then

$$\|S(t)z_0 - z_e\| \leq \frac{\varepsilon}{2}, \text{ for all } t \geq 0,$$

From (3), there is  $\hat{\delta}$ , with  $0 < \hat{\delta} < \delta$ , such that if  $\|z_0 - z_e\| \leq \hat{\delta}$ , then

$$\frac{\|\omega(z_e, z_0 - z_e)\|}{\|z_0 - z_e\|} \leq \frac{\varepsilon}{2}.$$

Then, from (9)

$$\|T_{z_e}(t)z_0 - z_e\| \leq \|\omega(z_e, z_0 - z_e)\| + \|S(t)z_0 - z_e\| \leq \varepsilon.$$

Thus,  $z_e$  is a stable equilibrium point of the linearization.  $\square$

The requirement in Theorem 3.3 that the linear system exhibits exponential stability is crucial. Below is an example due to Hans Zwart [16] illustrating this point. The example also highlights a fundamental difference between finite and infinite-dimensions; that is, exponential and asymptotic stability are not equivalent for linear systems in infinite-dimensions.

*Example 3.4:* Let  $\ell_2$  be the space of square summable sequences and  $\mathbb{N}$  the set of natural numbers with norm  $\|\cdot\|_{\ell_2}$ . For any  $z(t) = (z_1(t), z_2(t), \dots, z_n(t), \dots) \in \ell_2$  with  $n \in \mathbb{N}$ , consider

$$\dot{z}_n = -\frac{1}{n}z_n + z_n^2. \quad (10)$$

This system has infinitely many equilibrium  $z_e \in \ell_2$  since  $\dot{z}_n = 0$  if and only if  $-\frac{1}{n}z_n + z_n^2 = 0$  for  $n \in \mathbb{N}$ . This implies that  $z_n = 0, \frac{1}{n}$ . Therefore, the set of equilibria is

$$E = \left\{ z \in \ell_2 \mid z_n \in \left\{ 0, \frac{1}{n} \right\}, n \in \mathbb{N} \right\}.$$

Linearize the system (10) around  $z_e = 0$  to obtain

$$\dot{z}_n(t) = -\frac{1}{n}z_n(t), \quad t \geq 0 \quad (11)$$

which has solution

$$z(t) = (z_1(0)e^{-t}, z_2(0)e^{-\frac{1}{2}t}, \dots).$$

The linearized system (11) is asymptotically stable since

$$\begin{aligned} \lim_{t \rightarrow \infty} \|z(t) - z_e\|_{\ell_2} &= \lim_{t \rightarrow \infty} \|z(t)\|_{\ell_2} \\ &= \lim_{t \rightarrow \infty} \left( \sum_{n=1}^{\infty} z_n^2(0) e^{-\frac{2}{n}t} \right)^{\frac{1}{2}} \\ &= 0. \end{aligned}$$

Now consider the stability of the original nonlinear system (10). The solution to (10) is

$$z_n(t) = \frac{z_{0n}e^{-\frac{1}{n}t}}{z_{0n}n(-1 + e^{-\frac{1}{n}t}) + 1} \quad (12)$$

where  $z_{0n}$  is the initial condition. For any  $\delta > 0$ , choose  $n$  such that  $\frac{1}{n} < \delta$ . In the nonlinear system (10), choose components of the initial condition  $z_0$  to be zero except in the  $n^{\text{th}}$  position, which is chosen to be  $\frac{1}{n}$ ; that is,

$$z_0 = \left( 0, \dots, 0, \frac{1}{n}, 0, \dots \right).$$

Given this initial condition, the solution to (10) is

$$z(t) = \left( 0, \dots, \frac{1}{n}, \dots \right)$$

and hence  $\|z_0 - z_e\|_{\ell_2} = \frac{1}{n} < \delta$ . However,

$$\lim_{t \rightarrow \infty} \|z(t) - z_e\|_{\ell_2} = \frac{1}{n} \neq 0.$$

Hence, the zero equilibrium  $z_e$  to the nonlinear system (10) is not asymptotically stable.

Note that if the solution in (12) is truncated to  $N$  dimensions

$$\begin{aligned} \lim_{t \rightarrow \infty} \|z(t)\|_{\ell_2}^2 &= \lim_{t \rightarrow \infty} \sum_{n=1}^N \left| \frac{z_{0n}e^{-\frac{1}{n}t}}{z_{0n}n(-1 + e^{-\frac{1}{n}t}) + 1} \right|^2 \\ &= \sum_{n=1}^N \lim_{t \rightarrow \infty} \left| \frac{z_{0n}e^{-\frac{1}{n}t}}{z_{0n}n(-1 + e^{-\frac{1}{n}t}) + 1} \right|^2 = 0, \end{aligned}$$

and hence the approximated solution is asymptotically stable. The lack of stability is only apparent with the exact solution, not with the approximated solution.  $\square$

Since it is generally difficult to obtain a closed form representation of the semigroup, linearization and stability analysis is generally done using the generator  $F$  in (1). It is desirable to have conditions for linearized stability in terms of the generator. If the generator is Fréchet differentiable, it is not difficult to show that the semigroup generated by the linearization corresponds to the linearization of the original semigroup and Theorem 3.3 can be used. However, the generator in an infinite-dimensional space is typically unbounded and in these cases the generator is not Fréchet differentiable. Gâteaux differentiability is generally a more useful concept for linearization of the generator  $F$ .

*Definition 3.5:* Let  $F : \mathcal{D}(F) \subset X \rightarrow X$  be an operator defined on a Banach space  $X$ . The operator  $F$  is Gâteaux

differentiable at  $z_0 \in \mathcal{D}(F)$  if there exists a linear operator  $dF(z_0) : X \rightarrow X$  such that

$$\lim_{\varepsilon \rightarrow 0} \frac{F(z_0 + \varepsilon h) - F(z_0)}{\varepsilon} = dF(z_0)h,$$

where  $h, (z_0 + \varepsilon h) \in \mathcal{D}(F)$ .

The question is then, once the generator is linearized via a Gâteaux derivative, does the linearization generate a semigroup; and if so, does this semigroup correspond to the Fréchet derivative of the original system?

The situation for quasilinear systems is fairly well-understood. Consider a time-invariant quasilinear system on a Banach space  $X$ ,

$$\begin{aligned} \dot{z}(t) &= Az(t) + f(z(t)), \\ z(0) &= z_0, \end{aligned} \quad (13)$$

where  $z(t) \in X$  is the state and  $z_0$  is the initial condition. The operator  $A : \mathcal{D}(A) \subset X \rightarrow X$  is a linear operator that generates a  $C_0$ -semigroup on  $X$  and the nonlinear operator  $f : \mathcal{D}(f) \subset X \rightarrow X$  is Fréchet differentiable with  $Df(z)$  the Fréchet derivative of  $f$  at  $z$ . It is straightforward to show that  $A + Df(z)$  is the Gâteaux derivative of  $Az + f(z)$  at  $z$ . The linearized system corresponding to (13) at the equilibrium point  $z \in Z$  is

$$\frac{d\psi}{dt} = A\psi + Df(z)\psi \quad (14)$$

The following theorem is a special case of the more general result in [9, Thm. 2.1] for which the conditions are difficult to check. This theorem generalizes an earlier result [10, Theorem 11.22] which has more restrictive conditions on  $f$ .

**Theorem 3.6:** Consider equation (13). Suppose  $A$  generates a  $C_0$ -semigroup and  $f$  is Fréchet differentiable on  $X$ , and that the Fréchet derivative of  $f$  satisfies

$$\|Df(z_1) - Df(z_2)\| \leq c(r)\|z_1 - z_2\|,$$

for some  $r > 0$  and for all  $\|z_1\| \leq r, \|z_2\| \leq r$ , where  $c : [0, \infty) \rightarrow [0, \infty)$  is a continuous increasing function. Let  $z_e$  be an equilibrium point of (13). If  $A + Df(z_e)$  generates an exponentially stable semigroup, then  $z_e$  is a locally exponentially stable equilibrium point. Conversely, if the linearization is unstable, the original system is also unstable.

**Proof:** This is essentially shown in [9]. In section 3 of that paper, it is shown that the assumptions imply that the nonlinear semigroup is Fréchet differentiable at any equilibrium  $z_e$ , with generator  $A + dF(z_e)$ . In [9, Cor. 2.2] it is then shown that exponential stability of the linear semigroup implies local exponential stability of the original system, or Theorem 3.3 can be used. Theorem 3.3 implies instability of the original system if the linearization is unstable.

The assumptions on  $f$  in the following theorem are slightly different to those above.

**Theorem 3.7:** Let  $Z$  be a Hilbert space with norm  $\|\cdot\|_Z$  and inner product  $\langle \cdot, \cdot \rangle_Z$ . Consider the quasilinear equation in (13) and suppose it generates a semigroup,  $S(t)$ . Assume  $\text{Re}\langle Az, z \rangle_Z \leq 0$  for all  $z \in N_{z,r} = \{p \in Z : \|p - z\|_Z \leq$

$r\}$  and suppose  $f$  is Fréchet differentiable on  $N_{z,r}$  and its derivative,  $Df$ , is locally Lipschitz continuous on  $N_{z,r}$ . Also, for some positive constant  $K_{z,r}$  that depends on  $z$  and  $r$ , assume that

$$\sup_{\eta \in N_{z,r}} \|Df(\eta)\|_{op} = K_{z,r} < \infty$$

where  $\|\cdot\|_{op}$  is the operator norm. Then (14) generates the semigroup,  $T_z(t)$ , and

$$T_z(t) = DS(z_0)(t)$$

where  $DS(z_0)(t)$  is the Fréchet derivative of  $S(t)$  at  $z(0) = z_0$  for all  $0 \leq t \leq t_f$  for some positive  $t_f$ .

**Proof:** Since  $A$  generates a semigroup and  $Df(z)$  is bounded, then  $A + Df(z)$  generates a semigroup,  $T_z(t)$ , and  $\psi(t) = T_z(t)z_0$  is the unique solution to (14) [17, Theorem 3.2.1].

Let  $y, z \in N_{z,r}$  be solutions to (13) and define  $h(t) := y(t) - z(t)$ . Taking the time derivative of  $h$  and then the inner product with  $h$  leads to

$$\frac{1}{2} \frac{d}{dt} \|h\|_Z^2 = \text{Re}\langle Ah, h \rangle_Z + \text{Re}\langle f(y) - f(z), h \rangle_Z.$$

Since  $\text{Re}\langle Ah, h \rangle_Z \leq 0$  and applying the Cauchy-Schwarz inequality

$$\frac{1}{2} \frac{d}{dt} \|h\|_Z^2 \leq \|f(y) - f(z)\|_Z \|h\|_Z. \quad (15)$$

From the Mean Value Theorem,

$$\|f(y) - f(z)\|_Z \leq \sup_{z \in N_{z,r}} \|Df(z)\|_{op} \|y - z\|_Z.$$

and hence

$$\|f(y) - f(z)\|_Z \leq K_{z,r} \|y - z\|_Z \text{ for all } y, z \in N_{z,r}$$

since  $K_{z,r} = \sup_{\eta \in N_{z,r}} \|Df(\eta)\|_{op} < \infty$ . Equation (15) then becomes

$$\frac{d}{dt} \|h\|_Z^2 \leq 2K_{z,r} \|h\|_Z^2.$$

Integrating with respect to  $t$  yields

$$\|y(t) - z(t)\|_Z^2 \leq \|y(0) - z(0)\|_Z^2 e^{2K_{z,r}t}. \quad (16)$$

Define  $\phi(t) := y(t) - z(t) - \psi(t)$  where  $\psi(t)$  is the solution to (14). Taking the time derivative of  $\phi$  and then the inner product with  $\phi$  leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\phi\|_Z^2 &= \text{Re}\langle A\phi, \phi \rangle_Z + \text{Re}\langle Df(z)\phi, \phi \rangle_Z \\ &\quad + \text{Re}\langle f(y) - f(z) - Df(z)(y - z), \phi \rangle_Z. \end{aligned}$$

Since  $f$  is Fréchet differentiable, then  $\|Df(z)\|_{op}$  is bounded by a positive constant,  $M_z$ , and since the derivative of  $f$  is locally Lipschitz continuous, and  $\text{Re}\langle Az, z \rangle_Z \leq 0$ , then

$$\frac{1}{2} \frac{d}{dt} \|\phi\|_Z^2 \leq M_z \|\phi\|_Z^2 + \frac{L_{z,r}}{2} \|y - z\|_Z^2 \|\phi\|_Z$$

where  $L_{z,r}$  is the Lipschitz constant. Applying Gronwall's inequality [18, Lemma 2.8] with  $\phi(0) = 0$  implies

$$\|\phi\|_Z \leq L_{z,r} e^{2M_z t} \int_0^t e^{-2M_z s} \|y(s) - z(s)\|_Z^2 ds.$$

Applying equation (16) for  $t \in [0, t_f]$  with  $t_f$  any positive constant,

$$\|\phi\|_Z \leq L_{z,r} e^{2M_z t_f} \|y_0 - z_0\|_Z^2 \int_0^{t_f} e^{2(K_{z,r} - M_z)s} ds.$$

and solving the integrals leads to

$$\|\phi(t)\|_Z \leq k(t_f) \|y_0 - z_0\|_Z^2, \quad \text{for } t \in [0, t_f]$$

where

$$k(t_f) = \frac{L_{z,r}}{2(K_{z,r} - M_z)} (e^{2K_{z,r} t_f} - e^{2M_z t_f}).$$

It follows that

$$\|y - z - \psi\|_Z \leq k(t_f) \|y_0 - z_0\|_Z^2$$

and hence

$$\|S(t)y_0 - S(t)z_0 - T_z(t)\psi_0\|_Z \leq k(t_f) \|y_0 - z_0\|_Z^2.$$

Defining  $h_0 = y_0 - z_0$  leads to

$$\lim_{\|h_0\|_Z \rightarrow 0} \frac{\|S(t)(h_0 + z_0) - S(t)z_0 - T_z(t)h_0\|_Z}{\|h_0\|_Z} = 0$$

for all  $t \in [0, t_f]$ . More details can be found in [19, Theorem 2.23].  $\square$

Some limited results have been achieved for systems that are not quasilinear.

**Theorem 3.8:** [11, Section VI.8] Let  $Z$  be a Hilbert space with norm  $\|\cdot\|_Z$  and inner product  $\langle \cdot, \cdot \rangle_Z$ . Consider equation (13) with  $A : D(A) \rightarrow Z$  closed, negative and self-adjoint. Define  $Y = D((-A)^{1/2})$  with norm  $\|y\|_Y = \|(-A)^{1/2}y\|_Z$  and the dual space  $Y'$  with norm  $\|y\|_{Y'} = \|(-A)^{-1/2}y\|_Z$ . Assume that

$$f(z(t)) - f(w(t)) = L(z(t) - w(t)) + Q(z(t) - w(t))$$

where  $L$  is a linear bounded operator on  $Y$  to  $Y'$  such that for some  $0 < \epsilon \leq 1$  and positive constant  $c_\epsilon$  that depends on  $\epsilon$ ,

$$|\langle Lv, v \rangle_Z| \leq (1 - \epsilon) \|y\|_Y^2 + c_\epsilon \|y\|_Z^2$$

for all  $y \in Y$  and also assume  $Q$  satisfies

$$\|Q(z(t) - w(t))\|_{Y'} \leq k_1 \|z - w\|_Y^{1+\sigma_1}$$

for some  $k_1 > 0$  and  $\sigma_1 > 0$ . Also assume that for every  $R > 0$  there exists  $0 < \sigma_2 \leq 1$  and constant  $k_R$  depending on  $R$  such that

$$|\langle f(z) - f(w), z - w \rangle_Z| \leq k_R \|z - w\|_Z^{\sigma_2} \|z - w\|_V^{2-\sigma_2}$$

for all  $z, w \in Y$  with  $\|z\|_Z \leq R$  and  $\|w\|_Z \leq R$ . Given these conditions, the semigroup of (13) is Fréchet differentiable at any  $z$  with its derivative equal to the semigroup generated by  $A + df(z)$ .

Examples that satisfy the assumptions of Theorem 3.8 are found in [11]. These include special cases of the Navier-Stokes and wave equations.

The above classes are not exhaustive. Consider for example the Kuramoto-Sivashinsky (KS) equation [11], [20],

[21] with periodic boundary conditions defined on the Hilbert space  $L^2(-\pi, \pi)$

$$\frac{\partial z}{\partial t} + \nu \frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial x^2} + z \frac{\partial z}{\partial x} = 0, \quad t \geq 0,$$

$$\frac{\partial^n z}{\partial x^n}(-\pi, t) = \frac{\partial^n z}{\partial x^n}(\pi, t), \quad n = 0, 1, 2, 3, \quad (17)$$

$$z(x, 0) = z_0(x),$$

where  $z \in L^2(-\pi, \pi)$  is the state of the system,  $\nu > 0$  is the instability parameter,  $-\nu \frac{\partial^4 z}{\partial x^4}$  is the dissipative term,  $\frac{\partial^2 z}{\partial x^2}$  is the anti-dissipative term and  $z \frac{\partial z}{\partial x}$  is the nonlinear term [22]. This equation has a unique strong solution  $z(t) = S(t)z_0$ , where

$$z(t) \in L^2([0, T]; H_{per}^2(-\pi, \pi)) \cap L^\infty([0, T]; L^2(-\pi, \pi)),$$

and  $S(t)$  is a nonlinear  $C_0$ -semigroup [22, Theorem 5.4.3]. The stability analysis of the KS equation depends on the parameter  $\nu$ . If the instability parameter  $\nu > 1$ , the set of all constant equilibria is globally asymptotically stable. Furthermore, if  $\nu = 1$ , then the zero equilibrium is Lyapunov stable. This is proven using a Lyapunov function and LaSalle's invariance principle [14], [15],

Define  $A : \mathcal{D}(A) = H_{per}^4(-\pi, \pi) \subset H^4(-\pi, \pi) \rightarrow L^2(-\pi, \pi)$  by

$$Az = -\nu \frac{\partial^4 z}{\partial x^4} - \frac{\partial^2 z}{\partial x^2} \quad (18)$$

and the nonlinear operator  $J : \mathcal{D}(J) = H_{per}^1(-\pi, \pi) \subset H^1(-\pi, \pi) \rightarrow L^2(-\pi, \pi)$  by

$$J(z) = -z \frac{\partial z}{\partial x}. \quad (19)$$

The KS equation (17) can be written

$$\begin{aligned} \dot{z} &= Az + J(z), \\ z(0) &= z_0. \end{aligned} \quad (20)$$

The Gâteaux derivative  $dJ : H^1(-\pi, \pi) \subset L^2(-\pi, \pi) \rightarrow L^2(-\pi, \pi)$  of  $J$  at  $z_0$  is

$$dJ(z_0)z = \lim_{\epsilon \rightarrow 0} \frac{J(z_0 + \epsilon z) - J(z_0)}{\epsilon} = \frac{\partial}{\partial x}(z_0 z). \quad (21)$$

The linearized KS equation at  $z_0$  is

$$\dot{z} = (A - dJ(z_0))z. \quad (22)$$

The nonlinearity in the KS equation is not continuous and so the results for quasilinear systems do not apply. It also does not satisfy the assumptions of Theorem 3.8 as the linear operator in the KS equation is not negative and also the nonlinear operator does not satisfy the assumptions. However, the  $C_0$ -semigroup  $S(t)$  is Fréchet differentiable at any  $z_0 \in L^2(-\pi, \pi)$  and the derivative is the  $C_0$ -semigroup generated by the linearized KS equation at  $z_0$ . A similar result was shown in [11] but an additional assumption was required in the proof.

**Theorem 3.9:** [14], [15] Consider the nonlinear KS equation (20). The nonlinear semigroup  $S(t)$  is Fréchet differentiable at every  $z_0 \in L^2(-\pi, \pi)$ . Moreover, indicating the

Fréchet derivative of  $S$  by  $T$ ,  $A + dJ(z_0)$  is the generator of  $T$ .

The KS equation has an infinite number of equilibrium points. In particular, any constant function is an equilibrium to the KS equation. Define the closed invariant set of constant equilibria

$$Z_e = \{z_e : z_e \text{ is a constant function}\} \subset L^2(-\pi, \pi). \quad (23)$$

*Theorem 3.10:* Consider the KS equation (17). If the instability parameter  $\nu > 1$ , then any  $z_e \in Z_e$  is locally exponentially stable. If the instability parameter  $\nu < 1$ , then the KS equation is unstable.

**Proof.** Let  $z_e$  be any constant equilibrium. The operator  $(A - dJ(z_e))$  is a Riesz-spectral operator with eigenvalues  $\lambda_n = n^2(1 - \nu n^2) - inz_e$ , where  $n \in \mathbb{Z}$  [14, Theorem 5.2.1]. Since  $(A - dJ(z_e))$  is a Riesz-spectral operator, [17, Theorem 2.3.5 c] the spectrum determined growth assumption holds. The growth bound of the semigroup generated by  $A + dJ(z_e)$  is determined by the supremum of the real part of the eigenvalues. Hence, if  $\nu > 1$ , then all the eigenvalues of the linearized KS equation at a constant equilibrium have strictly negative real part, which results in a stable linearized system and if  $\nu < 1$ , then the linearized system is unstable.  $\square$

#### IV. CONCLUSIONS

The application of Lyapunov's indirect method requires the  $C_0$ -semigroup of the nonlinear system to be Fréchet differentiable. If the system linearized around an equilibrium is unstable or exponentially stable, then the equilibrium to the nonlinear system is unstable or locally exponentially stable, respectively. If the linearized system is only asymptotically stable, then Lyapunov's indirect method provides no conclusion about the stability of the equilibrium of the nonlinear system. This was illustrated by Example 3.4.

The key to using Lyapunov's Indirect Method is showing that the linearized generator corresponds to the generator of the Fréchet derivative of the original semigroup. Since the generator is generally unbounded, this is not straightforward. Fairly complete results are available for quasi-linear systems, however.

Partial differential equations that are not quasi-linear are also considered. The only general result is Theorem 3.8. There are many partial differential equations that do not fit this class. For example, the Kuramoto-Sivashinsky equation is not quasi-linear and does not satisfy the assumptions of Theorem 3.8. However, it has a Fréchet differentiable semigroup. Results using Lyapunov's Indirect Method for the stability of this equation are reviewed.

Further research to establish general results on differentiability of semigroups and their relation to the linearized generator is needed in order to expand the applicability of Lyapunov's Indirect Method.

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