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A switching observer for systems with linearizable error dynamics via singular time-scaling

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***Abstract*— In this paper, we propose a new observer design for nonlinear systems that can be linearized using a change of coordinates and a singular time re-scaling. Our observer is a switched system and the observer error dynamics are described, after time re-scaling, by a switched linear system that is uniformly exponentially stable. We also give necessary and sufficient conditions for linearizability via a change of coordinates and a singular time re-scaling. Our methods are illustrated by an example coming from the ABS literature.**

I. INTRODUCTION

The aim of an observer is to reconstruct, from the past values of the measured outputs, the states of a system for which a measure is not available [3]. For linear systems, the classical works of Kalman [15] and Luenberger [21] propose two well known approaches to solve the observation problem. The Luenberger observer has been extended to the case of nonlinear systems that can be linearized using a change of coordinates (see, e.g., [17], [18] and [22]) and to the class of uniformly observable systems (see, e.g., [5], [8] and [9]). More recent techniques (see, e.g., [16], [19], and [1]) have proposed still more flexibility in the observer design avoiding, for example, the necessity of considering high gains to counter nonlinear terms. Nevertheless, for systems that are not uniformly observable, the observation problem remains difficult because for them the observability properties depend on the control inputs that are applied to the system (see, e.g., [4] and [7]).

The aim of our paper is to propose an observer for a particular class of non-uniformly observable systems that are affine, with respect to the unmeasured states, up to a multiplication by a function of the output. Our work is strongly inspired by the time-rescaling approach proposed in [10] and [26], for the construction of observers for systems that cannot be linearized using a coordinate change only (see also [23]). The change of time scale considered in those works is regular and therefore the proposed observers can be constructed for uniformly observable systems only. In our approach, we handle the case of singular time-rescalings (and thus we deal with a class of of non-uniformly observable systems) and we solve the observer problem by transforming the observer error dynamics into a switched linear system, for which several stability analysis methods are available. In particular, the Lasalle-like approach of [11] can be used directly to prove the stability of the constructed observer.

To summarize, the novelty of our approach is two-fold. Firstly, we construct an observer for a class of non-uniformly observable systems (clearly, the convergence of the observer is assured for a certain family of distinguishing inputs only). Secondly, although the original system is continuous (smooth actually), we propose for it a switching asymptotically convergent observer.

In the second section we present a new observer design for the class of systems linearizable via a singular time-rescaling. We also state Theorem 1 that shows how to tune the observer gains. Section 3 of the paper analyzes the conditions that a nonlinear system must satisfy in order to be transformable into an affine system, with respect to the unmeasured states, up to a multiplication by a function of the output vanishing at the zero value of the output. Section 4 illustrates these ideas in the case of a simple estimation problem associated to ABS systems. The Appendix contains mathematical proofs.

II. SWITCHINGS AND SINGULAR OBSERVER DESIGN

In this paper, we consider systems of the following form

$$\dot{x} = f(x) + g(x)u, \quad y = h(x), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}$ is the measurement, and both f and g are C^∞-vector fields on $\mathbb{R}^n$. Among such systems, we have a particular interest for those that can be transformed, using a C^∞-diffeomorphism $z = \Phi(x)$, into the following affine (with respect to the unmeasured states) form

$$\dot{z} = s(y)\big(Az + d(y)\big) + b(y)u, \quad y = Cz, \tag{2}$$

where the pair (A, C) is observable and the vector fields b and d depend on the output only. Conditions for the existence of such a transformation are given in Section III.

Our interest for this class of systems comes from the possibility of building the observer

$$\frac{d\hat{z}}{dt} = s(y)\big(A\hat{z} + d(y) + K(s(y))(y - C\hat{z})\big) + b(y)u, \tag{3}$$

using a gain $K(\cdot)$ that depends on the system's output only. Defining the error $e = z - \hat{z}$, we obtain the observer error dynamics

$$\frac{de}{dt} = s(y)(A - K(s(y))C)e. \tag{4}$$

In the case of a non-vanishing function s (which we call the *regular* case), the analysis of the asymptotic stability of (4) has already been considered in [26]. In fact, if s is strictly positive, the observer gain $K(s(y))$ can be taken as a constant matrix K such that $A - KC$ is Hurwitz, which renders the error dynamics

$$\frac{de}{d\tau} = (A - KC)e$$

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asymptotically stable with respect to the new time-scale τ given by $\frac{d\tau}{dt} = s(y(t))$.

In this paper, our aim is to construct an observer that works even in the case when the function s might vanish (which we call the *singular* case). Throughout the function $s(y)$ is supposed strictly increasing and such that $s(y) = 0$ if and only if $y = 0$ and that $s'(0) \neq 0$. When the function $s(y(t))$ changes its sign, we are confronted with three problems. Firstly, on time-intervals when $s(y(t))$ is negative, we have to assure an (exponential) divergence of the error with respect to τ (and not its convergence). Secondly, we have to be sure that the time instances t_i at which $s(y(t_i)) = 0$ do not accumulate. Thirdly, system (2) is not uniformly observable. Indeed, it is easy to see that the control $u(t) \equiv 0$ renders indistinguishable any two states $z_0 \neq \tilde{z}_0$ such that $Cz_0 = C\tilde{z}_0 = 0$ (see Proposition 1 below).

The first problem will be solved by allowing the gain K to depend on $s(y)$ (actually on the sign of $s(y)$) whereas the second by ensuring the existence of a dwell time. Finally, the third problem by avoiding controls that make the system unobservable. In order to handle these three problems, the key step in our approach is to consider as a new time-scale

$$\tau(t) = \int_0^t |s(y(\nu))| d\nu. \tag{5}$$

This choice ensures that $d\tau/dt \geq 0$, independently of the values of $s(y)$.

We will assume that at any time instant t_i such that $y(t_i) = 0$, the control $u(t)$ law has been designed to achieve that the right-hand and left-hand derivatives $\frac{dy}{dt}(t_i^-)$ and $\frac{dy}{dt}(t_i^+)$ are nonzero (or, equivalently, that the limits $u(t_i^-)$ and $u(t_i^+)$ of $u(t)$ at t_i from the left and from the right, respectively, are nonzero). This choice of u is to cross the observability singularity $\{Cz = 0\}$ with a nonzero velocity and thus to render the system observable even at that singular locus (compare Proposition 1 below). A consequence of our assumptions is that the function w such that $\tau(t) = w(t)$ is globally invertible. We denote w^{-1} its inverse. Even if the dynamics (2) might be indefinite in the new time-scale τ when $y(t) = 0$, one can still define its output by taking $y(\tau) = y(w^{-1}(\tau))$.

In the new time-scale, if the observer gains are defined as

$$K(s(y(\tau))) = \begin{cases} K^+ & \text{if} : s(y(\tau)) > 0 \\ K^- & \text{if} : s(y(\tau)) < 0, \end{cases} \tag{6}$$

then the observer error dynamics (4) become

$$\frac{de}{d\tau} = \begin{cases} (A - K^+C)e & \text{if} : s(y(\tau)) > 0 \\ (-A + K^-C)e & \text{if} : s(y(\tau)) < 0. \end{cases} \tag{7}$$

This last system can be written in the form of a switched system

$$\frac{de}{d\tau} = A_{\sigma(\tau)} e, \tag{8}$$

by defining the switching signal $\rho(t) = \operatorname{sign}\big(y(t)\big)$ in the original time-scale, and transforming it into the new time as $\sigma(\tau) = \rho \circ w^{-1}(\tau)$.

Now, our aim is clear: we have to ensure the asymptotic stability of (7). In fact, we have transformed the analysis of the asymptotic stability of the origin of (4) into the stability analysis of (8), which is an autonomous switched linear system and numerous stability results are available (see, e.g., [20]) for that class of systems. Most of them are based on a classical Lyapunov-functions-based approach but in addition to them, LaSalle-like results have been proposed more recently [11]. In the latter, the stability properties of a switched linear system are proved via regularity assumptions on the set of switching signals.

Following [11], inside the set of arbitrary pairs $(e(\tau), \sigma(\tau))$, denote by $\mathcal{S}_{NC}$ the set of those that satisfy (8) and are such that e is piecewise differentiable and σ piecewise constant. Define, moreover, the set $\mathcal{S}[\tau_D] \subset \mathcal{S}_{NC}$, with $\tau_D > 0$, of pairs for which any two consecutive discontinuities of σ are separated by no less than τ_D. The constant τ_D is called the *dwell-time*. The origin of a switched system of the form (8) is said to be *uniformly exponentially stable* if there exists constants $a_0 > 0$ and $\lambda_0 > 0$ such that, for each $\tau \geq 0$, we have $\|e(\tau)\| \leq a_0 \exp(-\lambda_0 \tau)\|e(0)\|$. In this definition, the word *uniform* refers to the fact that a_0 and λ_0 do not depend on the switching signal, see [2].

Under a dwell-time condition, as a particular case of Theorem 4 of [11], one can prove that the switched linear system is uniformly exponentially stable if there exists a symmetric positive definite matrix that satisfies simultaneously the two non-strict Lyapunov equations below (10). The main result of this section, Theorem 1, shows that for our system (7) it is always possible to find a pair of gains K^+ and K^- such that this LaSalle-like condition is satisfied. The proof of Theorem 1 is detailed in the Appendix.

Recall that for any observable pair (A, C), there exist linear coordinates in which

$$A = \begin{pmatrix} a_1 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \dots & 0 & 1 \\ a_n & 0 & \dots & \dots & 0 \end{pmatrix}$$

and

$$C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}.$$

In these coordinates, the system is in *observer normal form*.

Theorem 1 *Consider an observable pair (A, C). Define $Q = C^T C$. For any given pair of gains K^+ and K^-, define*

$$A_+ = A - K^+C \quad \textit{and} \quad A_- = -A + K^-C. \tag{9}$$

If K^+ is such that A_+ is Hurwitz, then there exists a unique K^- such that the two following Lyapunov equations

$$A_+^T P + P A_+ = -Q \quad \textit{and} \quad A_-^T P + P A_- = -Q \tag{10}$$

admit a common solution P that is symmetric and positive definite.

Moreover, if (A,C) is in observer normal form, then the components k_i^- of K^- are expressed in terms of the components k_i^+ of K^+ by

$$k_i^- = (-1)^i k_i^+ + \big(1 - (-1)^i\big) a_i,$$

where a_i are the coefficients of the characteristic polynomial $p(\lambda) = \lambda^n - a_1\lambda^{n-1} - \cdots - a_{n-1}\lambda - a_n$ of A.

The following Theorem 2, which is the second main result of the paper and whose proof relies heavily on Theorem 1 and [11, Theorem 4]), shows that our observer converges under suitable conditions.

Firstly, in order to guarantee the system's observability, it is assumed that the pair (A, C) is observable and that the function $Cb(y)$ does not vanish at $y = 0$ (the relative degree of the system is 1). These assumptions are natural consequences of the following result summarizing observability properties of system (2).

Proposition 1 *(i) System (2) is observable if and only if the pair (A, C) is observable and the system has relative degree one at any z such that $Cz = 0$.*

(ii) System (2) is never uniformly observable. In fact, the control $u(t) \equiv 0$ renders indistinguishable any two states $z_0 \neq \tilde{z}_0$ such that $Cz_0 = C\tilde{z}_0 = 0$.

(iii) If the pair (A, C) is observable and the system has relative degree one at any z such that $Cz = 0$, then any control $u(t)$, for which there exists ϵ_u such that u is continuous on $[0, \epsilon_u[$ and $u(t) \neq 0$ for any $t \in [0, \epsilon_u[$, distinguishes all states $z \in \mathbb{R}^n$.

Secondly, in order to avoid control signals for which the observer does not converge, we have to guarantee the system's observability when crossing the singularity $\{y = 0\} = \{Cz = 0\}$, see Proposition 1. We thus impose some restrictions on the controllers that are used to govern the system. In order to describe these restrictions, some notations are necessary. Fix an open subset Ω_0 that contains the origin and fix a compact set Ω that contains Ω_0. Fix $\epsilon > 0$. Denote by Σ the intersection of Ω with the closed set $\{z \in \mathbb{R}^n : |Cz| \leq \epsilon\}$. By construction, the set Σ is compact. Define a_0, d_0, and s_0 as the maxima of the functions $|CAz|$, $|Cd(y)|$, and $|s(y)|$, respectively, on Σ. Define b_0 as the minimum of $Cb(y)$ on Σ. Introduce the constant $\alpha = s_0(a_0 + d_0)/b_0$. And, finally, fix $\beta > \alpha$.

We say that a controller is *admissible* for system (2) if it generates a control signal $u(\cdot)$ that satisfies together with its state trajectory $x(\cdot)$ the following properties:

(i) The control signal $u(t)$ is a piecewise continuous function of time such that $|u(t)| < \beta$, for $t \geq 0$;

(ii) On any given time interval $[a, b]$ such that the output satisfies $|y(t)| < \epsilon$, for $t \in [a, b]$, the sign of $u(t)$ is constant and $|u(t)| > \alpha$.

(iii) If $z(0) \in \Omega_0$ then $z(t) \in \Omega$, for each $t \geq 0$;

These assumptions, on the system's observability and on the class of admissible controllers, lead to the following result.

Theorem 2 *Consider system (2) and assume that the function $Cb(y)$ does not vanish at $y = 0$ and that the controller used to govern the system is admissible. If $z_0 \in \Omega_0$, then the switching signals $\rho(t)$ and $\sigma(\tau)$ generated by the controller admit a strictly positive dwell-time in the original and in the new time-scale, respectively.*

If, additionally, the pair (A, C) is observable and the gain matrices K^+ and K^- satisfy the conditions of Theorem 1 for the common solvability of the two coupled Lyapunov equations (10), then the origin of the observer's error dynamics (4) is asymptotically stable in the original time-scale associated to t and uniformly exponentially stable in the time-scale associated to τ.

Actually, in order to prove the uniform exponential stability of the origin, the dwell-time condition is not necessary (it is sufficient to have a persistent dwell-time [11, Theorem 4]). Nevertheless, since the conditions that are necessary for the well-posedness of the time-scale transformation lead to the existence of a dwell-time, we didn't consider in this paper a more general class of controllers that would lead only to a persistent dwell-time.

III. Conditions for singular linearizability

In this section we answer the question of when there exists a diffeomorphism $z = \Phi(x)$ bringing system (1) into (2) such that the pair (A, C) is observable and $s(0) = 0$ but $s'(0) \neq 0$.

Denote by $C^\infty(X_0)$ the ring of $\mathbb{R}$-valued C^∞-functions defined locally in a neighborhood X_0 of $x_0 \in \mathbb{R}^n$. For a function $\varphi \in C^\infty(X_0)$, denote by $Z(\varphi)$ the set of zeros of φ, that is,

$$Z(\varphi) = \{x \in X_0 \,:\, \varphi(x) = 0\}.$$

We will need the following technical result.

Lemma 1 *Consider $\varphi \in C^\infty(X_0)$ and $\psi \in C^\infty(X_0)$ such that $d\psi(x_0) \neq 0$. The following conditions are equivalent:*

(i) $\varphi = \alpha\psi$, for some $\alpha \in C^\infty(X_0)$;

(ii) $Z(\varphi) \subset Z(\psi)$;

(iii) The function $\alpha = \frac{\varphi}{\psi} \in C^\infty(X_0 \setminus Z(\psi))$ can be prolonged to a smooth function $\hat{\alpha} \in C^\infty(X_0)$, that is, there exists a function $\hat{\alpha} \in C^\infty(X_0)$ such that $\hat{\alpha}(x) = \alpha(x)$, for any $x \in X_0 \setminus Z(\psi)$;

(iv) If X_0' is a compact containing x_0 in its interior, then there exists $M \in \mathbb{R}$ such that $|\frac{\varphi}{\psi}(x)| < M$, for any $x \in X_0' \setminus Z(\psi)$.

Our characterization of systems of the form (1) equivalent to (2) will be given in three steps. In the first step, using Proposition 1 below, we will construct a new vector field $\hat{f}$ in which we get rid of the singularities of the vector field f (using a singular time re-scaling). In the second step, we will construct a vector field $\hat{g}$ using the observability rank condition of the pair $(\hat{f}, h)$; notice that the latter satisfies the observability rank condition although the original pair (f, h) does not, compare Proposition 1(ii). In the third step, to the couple of vector fields $(\hat{f}, \hat{g})$ we attach, via a regular time re-scaling, a couple $(\bar{f}, \bar{g})$. The required time re-scaling is computable in terms of solutions of an ordinary differential equation on the output space (see Remark 2 following Theorem 3). Our necessary and sufficient conditions are given in terms of the couple $(\bar{f}, \bar{g})$. Those conditions, as well as the construction of $(\bar{f}, \bar{g})$ follow from the main result of [26].

We will work around a fixed initial condition $x_0 \in \mathbb{R}^n$ and will assume that the diffeomorphism Φ satisfies $\Phi(x_0) = 0$.

Proposition 2 *If there exists a local diffeomorphism Φ transforming (1) into (2), then any among the equivalent* *conditions of Lemma 1 holds for all pairs $(\varphi, \psi) = (f_i, h)$, with $1 \leq i \leq n$, where f_i is the i-th component of f and h is the output function.*

If all pairs (f_i, h) of (1) satisfy the equivalent conditions of Lemma 1, then, due to condition (iii), each $\frac{f_i}{h} = \alpha$ can be extended to a C^∞-function $\hat{\alpha}$, which we will denote $\hat{f}_i$. Define the C^∞-vector field

$$\hat{f} = \hat{f}_1 \frac{\partial}{\partial x_1} + \cdots + \hat{f}_n \frac{\partial}{\partial x_n}.$$

A necessary condition for (1) to be transformed via $z = \Phi(x)$ into into (2) is that the pair $(\hat{f}, h)$ satisfies the following local observability rank condition (see, e.g., [14] and [24])

$$\dim \operatorname{span}\{dh, dL_{\hat{f}}h, \ldots, dL_{\hat{f}}^{n-1}h\}(x_0) = n.$$

In that case, following [17] (see also [18]) define a vector field $\hat{g}$ (a "dummy" input) by

$$L_{\hat{g}} L_{\hat{f}}^j h = \begin{cases} 0 & \text{if } 0 \leq j \leq n-2 \\ 1 & \text{if } j = n-1. \end{cases} \tag{11}$$

For $j \geq 2$ we put $l_j = \frac{j(j-1)}{2} + 1$. In order to avoid the trivial case, we will assume throughout $n \geq 2$. We have the following result.

Theorem 3 System (1) *is, locally around x_0, equivalent under a diffeomorphism $z = \Phi(x)$ to the system* (2)*, where* $s(0) = 0$ *and* $s'(0) \neq 0$*, if and only if in a neighborhood of x_0 all pairs $(\varphi, \psi) = (f_i, h)$, where f_i is the i-th component of f and h is the output function, fulfill the equivalent conditions of Lemma 1 and the vector fields $\hat{f}$ and $\hat{g}$ satisfy*

(i) the pair $(\hat{f}, h)$ satisfies at x_0 the local observability rank condition;

(ii) $dL_{\hat{g}} L_{\hat{f}}^n h = l_n \lambda dL_{\hat{f}} h \bmod \operatorname{span}\{dh\}$*, for some smooth function λ;*

(iii) $\left[ad_{\bar{f}}^i \bar{g}, ad_{\bar{f}}^j \bar{g}\right] = 0$*, for $0 \leq i < j \leq n-1$, where $\bar{f} = \frac{1}{\bar{s}} \hat{f}$, $\bar{g} = \bar{s}^{n-1} \hat{g}$, and $\bar{s} = \exp \theta$, with θ being a solution of*

$$L_{ad_{\bar{f}}^j \hat{g}} \theta = \begin{cases} 0 & \text{if } 0 \leq j \leq n-2 \\ (-1)^{n-1} \lambda & \text{if } j = n-1. \end{cases}$$

(iv) $\left[g, ad_{\bar{f}}^j \bar{g}\right] = 0$*, for $0 \leq j \leq n-2$.*

Moreover, $d(0) = 0$ if and only if $\hat{f}(x_0) = 0$.

Remark 1 The time re-scaling $s(y)$ is actually a product of two transformations $s(y) = y\bar{s}(y)$. Indeed, the dynamics to be linearized are obtained by, firstly dividing f by $h = y$ which gives $\hat{f}$ (a singular re-scaling) and then by dividing $\hat{f}$ by $\bar{s} = \exp \theta$ (a regular re-scaling) yielding $\bar{f}$.

Remark 2 The regular time re-scaling $\bar{s} = \exp \theta$ is defined by the ordinary differential equation

$$\frac{d\theta(y)}{dy} = \lambda(y)$$

on the output space $\mathbb{R}$. Hence it follows that

$$s(y) = y\bar{s}(y) = y \exp\left(\int_0^y \lambda(v) dv\right).$$

IV. Simulations on a simple example

In this section, we illustrate Theorems 1 and 2 with the design of an extended braking stiffness (XBS) observer, a problem that has its origins in the anti-lock brake systems literature (see, e.g., [12] and the references therein).

During a straight-line ABS braking manoeuvre, the dynamics of the wheel can be described [12] using the model

$$\begin{aligned} \frac{dz_1}{dt} &= \frac{-a}{v_x(t)} z_1 z_2 + bu \\ \frac{dz_2}{dt} &= (cz_2 + z_3) \frac{z_1}{v_x(t)} \qquad \text{and} \qquad y = z_1, \\ \frac{dz_3}{dt} &= 0 \end{aligned} \tag{12}$$

where a, b, and c are known constants[1] The longitudinal speed of the vehicle (which is strictly positive during braking manoeuvres) is denoted v_x and the control u is the time-derivative of the brake pressure. The state z_1 is the wheel acceleration offset (the difference between the acceleration of the wheel and that of the vehicle), which is the only measurable variable. The state z_2 is the XBS, which is the derivative of the tyre characteristics (described by Burckhardt's model [6]). The state z_3 is a constant that depends on the parameters of Burckhardt's model. The reader is referred to [12] for a more detailed description of this model.

In its initial form, system (12) does not belong to the class of autonomous systems considered in the previous sections. Nevertheless, if we consider the regular time re-scaling $\frac{dt'}{dt} = \frac{1}{v_x(t)}$ and the new control $\tilde{u}(t) = u(t)v_x(t)$, then system (12) with respect to the new time t' is of the form (2). In order to keep a coherent notation with previous sections, in what follows, despite of the abuse of notation this new time scale is again denoted by t and the new control by u.

Our system is already in the form (2), so after the singular time re-scaling $d\tau = s(y(t))dt$, where $s(y) = -ay = -az_1$, and in the new coordinates $\bar{z}$, where $\bar{z}_1 = z_1$, $\bar{z}_2 = z_2 + \dfrac{c}{a} z_1$, and $\bar{z}_3 = -\dfrac{z_3}{a}$, the system is in the observer normal form. In these coordinates, we get the following observer error dynamics with respect to the new time-scale τ

$$\frac{de}{d\tau} = \begin{cases} A_+ e = \begin{pmatrix} -\frac{c}{a} - k_1^+ & 1 & 0 \\ -k_2^+ & 0 & 1 \\ -k_3^+ & 0 & 0 \end{pmatrix} e & \text{if } s(y) > 0 \\ A_- e = \begin{pmatrix} \frac{c}{a} + k_1^- & -1 & 0 \\ k_2^- & 0 & -1 \\ k_3^- & 0 & 0 \end{pmatrix} e & \text{if } s(y) < 0. \end{cases} \tag{13}$$

If we want A_+ to be Hurwitz, the gain K^+ must satisfy

$$\begin{aligned} & k_1^+ > -\frac{c}{a} \\ & k_2^+ > 0 \\ & 0 < k_3^+ < \left(\frac{c}{a} + k_1^+\right) k_2^+. \end{aligned} \tag{14}$$

[1] We consider here that the road conditions are known. The problem of unknown road conditions has been considered recently in [13], and leads to a four-dimensional model of the form (2) for which Theorem 1 can be applied (like for the simpler three-dimensional model considered here).

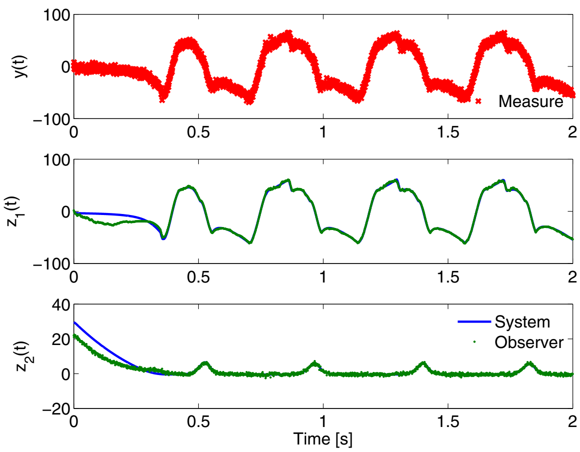

Fig. 1. Simulation of a five-phase hybrid ABS algorithm [12] with a closed-loop wheel acceleration control based on our singular observer.

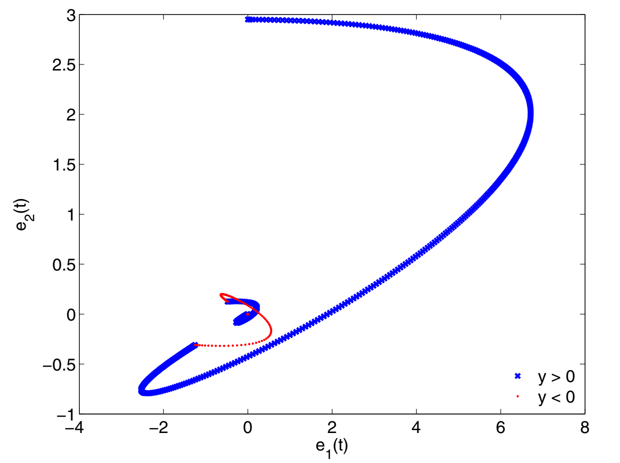

Fig. 2. Phase-plane evolution of the previous simulation, obtained by a projection of the three-dimensional error on the (e_1, e_2) plane.

Since the coordinates are those of the observer normal form, Theorem 1 gives directly the value of K^- that has to be used in the observer

$$\begin{aligned} k_1^- &= -k_1^+ - 2\frac{c}{a} \\ k_2^- &= k_2^+ \\ k_3^- &= -k_3^+. \end{aligned} \tag{15}$$

If we want, for example, the characteristic polynomial of A_+ to have a triple pole $-r$, for the following numerical values

$$\begin{aligned} a &= 187.5\ (N.kg^{-1}),\ b = -4.4\ (N.m.bar^{-1}kg^{-1}), \\ c &= 24,\ v_x = 17.8\ (m/s),\ \text{ and } r = 0.1, \end{aligned}$$

we obtain, for example solving the Lyapunov equations of Theorem 1 with Matlab, two matrices P_+ and P_- whose equality is guaranteed by the theorem.

In order to apply Theorem 2, one must consider a control law that satisfies its conditions. A possible choice is the hybrid five-phase ABS described in [25], which satisfies all conditions required for a control to be admissible. With this control law, our switching observer (13) is thus uniformly exponentially stable in the new time-scale and asymptotically stable in the original time. The aim of our simulation is to illustrate this property. The simulation results are shown on Figures 1 and 2. At the time instants where the road conditions change (here $t = 0$), the observer errors are different from 0. Nevertheless, they converge asymptotically to 0 if the road conditions remain constant.

Appendix

In order to prove Theorem 1, we will state and prove two claims and a lemma. We will need, moreover, an additional concept. Define $y_+(t, x_0^+)$ and $y_-(t, x_0^-)$, as the solutions of systems

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} &=& A_+ x \\ y &=& Cx \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{rcl} \dfrac{dx}{dt} &=& A_- x \\ y &=& Cx, \end{array} \right.$$

respectively, that start at the initial condition $x^\pm(0) = x_0^\pm$.

Claim 1 *Assume that the pairs (A_+, C) and (A_-, C) are both observable. The two following conditions are equivalent:*

(i) There exists a matrix S such that, for any initial condition $x_0^+ \in \mathbb{R}^n$, the outputs of the two systems satisfy $y_-(t, Sx_0^+) = y_+(t, x_0^+)$.

(ii) The spectra of A_+ and A_- coincide.

Moreover, if S exists then it is unique.

Proof: The outputs of the systems

$$y_\pm(t, x_0^\pm) = Ce^{tA_\pm} x_0^\pm \tag{16}$$

are analytic because the exponential function is analytic. Therefore the two outputs are equal if and only if their time-derivatives at zero, of any order $k \geq 0$, coincide

$$y_+^{(k)}(0, x_0^+) = y_-^{(k)}(0, x_0^-). \tag{17}$$

Thus a matrix S such that $y_-(t, Sx_0^+) = y_+(t, x_0^+)$ exists if and only if

$$y_+^{(k)}(0, x_0^+) = y_-^{(k)}(0, Sx_0^+) \tag{18}$$

or, equivalently,

$$CA_+^k x_0^+ = CA_-^k S x_0^+, \quad \text{for } k \geq 0. \tag{19}$$

Since this is true for any $x_0^+ \in \mathbb{R}^n$, we conclude that $y_-(t, Sx_0^+) = y_+(t, x_0^+)$ if and only if

$$CA_+^k = CA_-^k S, \quad \text{for } k \geq 0. \tag{20}$$

Define

$$M_+ = \begin{pmatrix} C \\ CA_+ \\ \vdots \\ CA_+^{n-1} \end{pmatrix} \quad \text{and} \quad M_- = \begin{pmatrix} C \\ CA_- \\ \vdots \\ CA_-^{n-1} \end{pmatrix}. \tag{21}$$

The pairs (A_+, C) and (A_-, C) are observable hence rank (M_+) = rank $(M_-) = n$. The equality (20) implies

$$M_+ = M_- S. \tag{22}$$

$(i) \implies (ii)$ Assume that a matrix S, such that $y_-(t, Sx_0^+) = y_+(t, x_0^+)$, exists and thus (20) holds. In particular, we have $M_+ = M_- S$.

The characteristic polynomial of A_+ is defined as

$$p(\lambda) = \lambda^n + p_{n-1}^+ \lambda^{n-1} + \ldots + p_1^+ \lambda + p_0^+, \quad (23)$$

where p_k^+, for $0 \le k \le n-1$, are its coefficients. Substituting the matrix A_+ for λ, the Cayley-Hamilton theorem states that $p(A) = 0$. Thus,

$$A_+^n = -p_{n-1}^+ A_+^{n-1} - \ldots - p_1^+ A_+ - p_0^+. \quad (24)$$

Multiplying this equation by C on the left, we obtain

$$CA_+^n = -p_{n-1}^+ CA_+^{n-1} - \ldots - p_1^+ CA_+ - p_0^+ C. \quad (25)$$

Applying analogous computations for A_- and multiplying both sides by S gives

$$CA_-^n S = -p_0^- CS - p_1^- CA_- S - \ldots - p_{n-1}^- CA_-^{n-1} S, \quad (26)$$

where p_k^-, with $0 \le k \le n-1$, are the coefficients of the characteristic polynomial of A_-.

Define $p^\pm = (p_0^\pm, p_1^\pm, \ldots, p_{n-1}^\pm)$. We have $CA_+^n = CA_-^n S$ (because of (20) applied for $k = n$), hence (25) and (26) imply $M_+ p^+ = M_- S p^-$. The equality $M_+ = M_- S$ gives $M_+(p^+ - p^-) = 0$ yielding $p^+ = p^-$, that is,

$$p_k^+ = p_k^-, \quad 0 \le k \le n-1. \quad (27)$$

This implies that the spectra of A_+ and A_- coincide.

$(ii) \implies (i)$ Define S by $S = M_-^{-1} M_+$. In order to prove that $y_-(t, Sx_0^+) = y_+(t, x_0^+)$ it is enough, as we have observed, to show that

$$CA_+^k = CA_-^k S, \quad \text{for } k \ge 0. \quad (28)$$

The equality (28) holds, for $0 \le k \le n-1$, by the definition of S.

Now, we suppose that (28) holds, for $0 \le k \le n+i$ and a fixed $i \ge -1$. We will show that $CA_+^{n+i+1} = CA_-^{n+i+1} S$. Due to the Cayley-Hamilton theorem, the vector CA_+^n is expressed by (25). Multiplying this equation by A_+^{i+1} on the right, we obtain

$$CA_+^{n+i+1} = -p_{n-1}^+ CA_+^{n+i} - \ldots - p_0^+ CA_+^{i+1}. \quad (29)$$

Similarly, the vector CA_-^n can be expressed as

$$CA_-^n = -p_{n-1}^+ CA_-^{n-1} - \ldots - p_1^+ CA_- - p_0^+ C. \quad (30)$$

Multiplying this equation by $A_-^{i+1} S$ on the right, we obtain

$$CA_-^{n+i+1} S = -p_{n-1}^+ CA_-^{n+i} S - \ldots - p_0^+ CA_-^{i+1} S. \quad (31)$$

The spectra of A_+ and A_- coincide and thus so do they characteristic polynomials. Therefore, by the induction assumption, the right-hand sides of (29) and (31) are equal and thus

$$CA_+^{n+i+1} = CA_-^{n+i+1} S.$$

By mathematical induction, (28) holds for any $k \ge 0$ implying that $y_-(t, Sx_0^+) = y_+(t, x_0^+)$.

To prove uniqueness of S, observe that $M_+ = M_- S$ implies $S = M_-^{-1} M_+$, which represents S in a unique way. ■

The following result is a direct consequence of observability.

Claim 2 *Assume that the pair (A, C) is observable. Then, for any observer gain vector K^+, there exists a unique gain K^- such that the spectra of A_+ and A_- coincide.*

Lemma 2 *Assume that the pair (A, C) is observable. We have:*

(i) For any observer gain vector K^+, there exists a unique gain K^- and a unique matrix S such that $y_-(t, Sx_0) = y_+(t, x_0)$.

(ii) If, moreover, the coordinates x are such that the pair (A, C) is in observer form then

$$S = \operatorname{diag}\big((-1)^0, (-1)^1, (-1)^2, \cdots, (-1)^{n-2}, (-1)^{n-1}\big)$$

and

$$k_i^- = (-1)^i k_i^+ + \big(1 - (-1)^i\big) a_i.$$

Proof: [of Lemma 2] (i) The point (i) is obtained directly, by applying first Claim 2 (which is a standard fact of control theory) and then Claim 1.

(ii) Define the gain matrices

$$K^+ = \begin{pmatrix} k_1^+ & \ldots & k_n^+ \end{pmatrix}^T \text{ and } K^- = \begin{pmatrix} k_1^- & \ldots & k_n^- \end{pmatrix}^T. \quad (32)$$

Since (A, C) is in observer normal form, the matrices $A_\pm$ have the following form

$$A_+ = \begin{pmatrix} a_1 - k_1^+ & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \ldots & 0 & 1 \\ a_n - k_n^+ & 0 & \ldots & \ldots & 0 \end{pmatrix}$$
$$\text{and } A_- = \begin{pmatrix} -a_1 + k_1^- & -1 & 0 & \ldots & 0 \\ \vdots & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & 0 & \ldots & 0 & -1 \\ -a_n + k_n^- & 0 & \ldots & \ldots & 0 \end{pmatrix}. \quad (33)$$

The characteristic polynomial of the matrix A_+ is

$$\begin{aligned} \det(I\lambda - A_+) &= (k_n^+ - a_n) + (k_{n-1}^+ - a_{n-1})\lambda \\ &\quad + \ldots + (k_1^+ - a_1)\lambda^{n-1} + \lambda^n. \end{aligned} \quad (34)$$

and the characteristic polynomial of the matrix A_- is

$$\begin{aligned} \det(I\lambda - A_-) &= (-1)^n (k_n^- - a_n) + \ldots \\ &\quad + (-1)^1 (k_1^- - a_1)\lambda^{n-1} + \lambda^n. \end{aligned} \quad (35)$$

Since, by Claim 2, the spectra of A_+ and A_- are equal (which means that the characteristic polynomials of A_+ and A_- coincide), we obtain that (34) and (35) coincide. Therefore,

$$k_i^- = (-1)^i k_i^+ + (1 - (-1)^i) a_i. \quad (36)$$

Now, it remains to compute the matrix S. Define

$$S = \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ s_{n1} & s_{n2} & \dots & s_{nn} \end{pmatrix}. \tag{37}$$

We cannot use directly the equation (22) to compute the matrix S, because the computation of the matrices M^+ and M^- is complicated. We know, however, that each equation from (20) will help us to identify the corresponding line of the matrix S. We denote by e_j, for $1 \leq j \leq n$, the j-th line of the identity matrix of order n.

- Indeed, from $C = CS$, we obtain
$$\begin{pmatrix} s_{11} & \dots & s_{1n} \end{pmatrix} = e_1. \tag{38}$$
- From $CA_+ = CA_- S$, we obtain
$$-(k_1^+ - a_1)C + e_2 = (k_1^- - a_1)CS - e_2 S. \tag{39}$$
Using (36), for $i = 1$ and $C = CS$, we get
$$\begin{pmatrix} s_{21} & \dots & s_{2n} \end{pmatrix} = -e_2. \tag{40}$$
- From $CA_+^2 = CA_-^2 S$, we obtain
$$\begin{aligned} & -(k_1^+ - a_1)CA_+ - (k_2^+ - a_2)C + e_3 \\ = \quad & (k_1^- - a_1)CA_- S - (k_2^- - a_2)CS + e_3 S. \end{aligned} \tag{41}$$
Because of equation (36), for $1 \leq i \leq 2$, and $CA_+^k = CA_-^k S$, for $0 \leq k \leq 1$, we conclude
$$\begin{pmatrix} s_{31} & \dots & s_{3n} \end{pmatrix} = e_3. \tag{42}$$
- We continue analogous computations for all equations of (20), up to
$$CA_+^{n-1} = CA_-^{n-1} S.$$
We obtain
$$\begin{aligned} & -(k_1^+ - a_1)CA_+^{n-2} - \dots - (k_{n-2}^+ - a_{n-2})CA_+ \\ & -(k_{n-1}^+ - a_{n-1})C + e_n \\ = \quad & (-1)^0(k_1^- - a_1)CA_-^{n-2}S + \dots + \\ & (-1)^{n-2}(k_{n-1}^- - a_{n-1})CS + (-1)^{n-1}e_n S. \end{aligned} \tag{43}$$
As a consequence of equation (36) and $CA_+^k = CA_-^k S$, for $0 \leq k \leq n-2$, we conclude
$$\begin{pmatrix} s_{n1} & \dots & s_{nn} \end{pmatrix} = (-1)^{n-1} e_n. \tag{44}$$

Therefore the matrix S is proven to be

$$S = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & (-1)^{n-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & (-1)^{n-1} \end{pmatrix}, \tag{45}$$

or, equivalently,

$$S = \mathrm{diag}\big((-1)^0, (-1)^1, \cdots, (-1)^{n-2}, (-1)^{n-1}\big).$$

■

Proof: [of Theorem 1] Since (A, C) is observable, the pairs (A_+, C) and (A_-, C) are also observable. Moreover, by Claim 1, the matrices A_+ and A_- are Hurwitz. When the matrices A_+ and A_- are Hurwitz, it is well known that each equation $A_\pm^T P + PA_\pm = -Q$ has always a unique solution, which is symmetric positive definite when (A, C) is observable, even if the matrix Q is just positive semi-definite. One possible way to describe P_+ is the following

$$P_+ = \int_0^\infty (e^{tA_+})^T C^T C e^{tA_+} dt. \tag{46}$$

Thus,

$$\begin{aligned} x_0^+ P_+ x_0^+ &= \int_0^\infty (Ce^{tA_+}x_0^+)^T (Ce^{tA_+}x_0^+) ds \\ &= \int_0^\infty (y_+(t, x_0^+))^T y_+(t, x_0^+) dt \\ &= \int_0^\infty ||y_+(t, x_0^+)||^2 dt, \end{aligned} \tag{47}$$

where the output $y_+(t, x_0^+) = Ce^{tA_+}x_0^+$.
Similarly, we also have

$$P_- = \int_0^\infty (e^{tA_-})^T C^T C e^{tA_-} dt. \tag{48}$$

Then,

$$(Sx_0^+)^T P_- (Sx_0^+) = \int_0^\infty ||y_-(t, x_0^+)||^2 dt, \tag{49}$$

where the output $y_-(t, Sx_0^+) = Ce^{tA_-}Sx_0^+$.

By Lemma 2, we know that, for any observer gain vector K^+, there exist a unique gain vector K^- and a unique matrix S such that $y_-(t, Sx_0^+) = y_+(t, x_0^+)$. We then obtain

$$(x_0^+)^T S^T P_- S x_0^+ = (x_0^+)^T P_+ x_0^+, \quad \forall x_0^+. \tag{50}$$

Therefore,

$$S^T P_- S = P_+. \tag{51}$$

If (A, C) is in observer normal form, by Lemma 2, the matrix S is given by (45). Thus,

$$\begin{aligned} & S^T P_- S = P_+ \\ \Leftrightarrow \quad & p_{ij}^+ = \begin{cases} p_{ij}^- & \text{if } i + j = 0 \mod 2 \\ -p_{ij}^- & \text{if } i + j \neq 0 \mod 2, \end{cases} \end{aligned} \tag{52}$$

where $p_{ij}^+,\ p_{ij}^-$ are, respectively, the elements of the i-th line and j-th column of the matrices $P_+,\ P_-$.

Now we will prove that $P_+ = P_-$. In the observer normal form, the matrix A_+ is described by (33). Clearly, we have the following relation

$$A_+ e_i = e_{i-1}, \quad \text{for } 2 \leq i \leq n, \tag{53}$$

where e_i, with $1 \leq i \leq n$, are the columns of the identity matrix of order n.

Now, multiplying the Lyapunov equation $A_+^T P + PA_+ = -Q$ by e_i^T on the left , and by e_j on the right , for $2 \leq i, j \leq n$, we obtain

$$e_i^T (A_+^T P + PA_+) e_j = -e_i^T Q e_j. \tag{54}$$

Using the relation (53), we conclude

$$\begin{aligned} & e_{i-1}^T P e_j + e_i^T P e_{j-1} = -e_i^T Q e_j \\ \Leftrightarrow \quad & p^+_{(i-1)j} + p^+_{i(j-1)} = -q_{ij} \\ \Leftrightarrow \quad & p^+_{(i-1)j} + p^+_{i(j-1)} = 0, \quad \text{for } 2 \le i, j \le n, \end{aligned} \tag{55}$$

because $q_{ij} = 0$, for $2 \le i, j \le n$. This condition implies that the sum of the two adjacent elements on the antidiagonal and on other parallel lines with the anti-diagonal are always equal to 0. Therefore, all the elements on the anti-diagonal, and on any line parallel to the anti-diagonal, are equal in absolute value. If the number of elements on the anti-diagonal and on the other parallel lines with the anti-diagonal is even, then we have $p^+_{ij} = -p^+_{ji}$. However, since the matrix P_+ is symmetric, it follows that $p^+_{ij} = -p^+_{ji} = 0$. If the number of elements on the anti-diagonal and on the other parallel lines with the anti-diagonal is odd, then $p^+_{ij} = p^+_{ji}$. Finally, we obtain

$$p^+_{ij} = \begin{cases} p^+_{ji} = p^+_{ii} & \text{if } i + j = 0 \mod 2 \\ & \text{and } |i - j| = 0 \mod 4 \\ p^+_{ji} = -p^+_{ii} & \text{if } i + j = 0 \mod 2 \\ & \text{and } |i - j| = 2 \mod 4 \\ -p^+_{ji} = 0 & \text{if } i + j \neq 0 \mod 2 \ . \end{cases} \tag{56}$$

Similarly, the matrix A_- has a following property

$$A_- e_i = -e_{i-1}, \quad \text{for } 2 \le i \le n. \tag{57}$$

With similar computations and arguments, we obtain

$$p^-_{ij} = \begin{cases} p^-_{ji} = p^-_{ii} & \text{if } i + j = 0 \mod 2 \\ & \text{and } |i - j| = 0 \mod 4 \\ p^-_{ji} = -p^-_{ii} & \text{if } i + j = 0 \mod 2 \\ & \text{and } |i - j| = 2 \mod 4 \\ -p^-_{ji} = 0 & \text{if } i + j \neq 0 \mod 2 \ . \end{cases} \tag{58}$$

Combining (52), (56) and (58), we obtain

$$p^+_{ij} = \begin{cases} p^-_{ij} = p_{ii} & \text{if } i + j = 0 \mod 2 \\ & \text{and } |i - j| = 0 \mod 4 \\ p^-_{ij} = -p_{ii} & \text{if } i + j = 0 \mod 2 \\ & \text{and } |i - j| = 2 \mod 4 \\ p^-_{ij} = 0 & \text{if } i + j \neq 0 \mod 2 \ . \end{cases} \tag{59}$$

or,

$$P_+ = P_- = \begin{pmatrix} p_{11} & 0 & -p_{22} & 0 & \dots \\ 0 & p_{22} & 0 & \dots & \dots \\ -p_{22} & 0 & \ddots & \dots & \dots \\ 0 & \dots & \dots & \ddots & \dots \\ \vdots & \vdots & \vdots & \vdots & p_{nn} \end{pmatrix}. \tag{60}$$

■

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