21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Instability of Interconnected Positive Real Systems

Masaki Inoue[1], Jun-ichi Imura[2], and Takayuki Ishizaki[2]

Abstract— **In this paper, we study instability analysis of a feedback system composed of a nominal linear system and a positive real uncertainty. We propose a new definition of generalized strictly positive realness to show that it is sufficient for preserving the same instability property in the feedback system as that in the nominal system for all uncertainties. Then, an illustrative example for robustness analysis of a genetic oscillator model with positive real uncertainties is presented. Finally, we discuss an implication of the derived instability condition from the viewpoint of stability and instability analysis of a positive real network system, i.e., a network system composed of positive real subsystems that are internally connected in a specific network topology. We show merits and demerits of such network construction strategy for global stability to be guaranteed.**

I. INTRODUCTION

A purpose of systems biology is to reveal the cause and mechanism for the robustness of complex behavior appeared in biomolecular network systems, such as oscillatory and switching behavior [1], [2], [3], [4], [5]. The researchers in the research field attempt to characterize and qualitatively evaluate the robustness of a simplified oscillator model derived from an actual complex biological system. The robustness is against uncertainties such as modelling errors. For example, anticipated gene affection, which are ignored in modelling process, affect a nominal gene expression network as illustrated in Fig. 1.

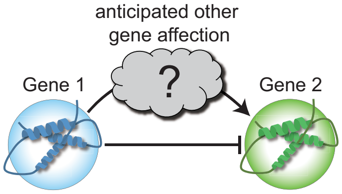

Fig. 1. Uncertainty in gene expression

An approach to the robustness analysis for oscillatory behavior is to evaluate the instability of an equilibrium, which is necessary for the existence of oscillation for a class of nonlinear systems [1], [3], [4], [5], [6], [7], [8], [9]. The robustness of the existence of oscillatory behavior is identified with that of the instability property of an equilibrium. The robustness of the instability is evaluated against parametric perturbations [1], [3], [4], [5], [6], [7], [8] as well as dynamic perturbations [9], [10], [11]. The latter analysis is based on a small-gain-type instability condition derived by the authors. However, such gain-based analysis is not always compatible with and valid for the analysis of biomolecular models since connection between two types of genes is represented by a Hill function, which can have high-gain, and therefore we tend to conclude that a biomolecular model is fragile provided that the robustness is measured by its gain. We know that biomolecular models have some kind of monotonicity in their input-output maps as well. By considering such a monotonicity property, we need further instability analysis methods, which can be applicable to analysis of systems with high-gain uncertainties.

In this paper, we assume that the uncertainty belongs to the set of positive real systems and consider instability analysis of a feedback system composed of a nominal system and such an uncertainty. We propose a new definition of generalized strictly positive realness to show that it is sufficient for preserving the same instability property in the feedback system as that in the nominal system for all uncertainties. Then, an illustrative example for robustness analysis of a genetic oscillator model [1], [7], [9] with positive real uncertainties is presented. Finally, we discuss an implication of the derived instability condition from the viewpoint of stability and instability analysis of a positive real network system, i.e., a network system composed of positive real subsystems which are internally connected in a specific network topology. We raise a question for merits and demerits of such network construction strategy for global stability to be guaranteed.

Notation and Definition: The symbol $\mathcal{RH}_\infty$ is the space that consists of all proper and complex rational stable transfer function matrices. The poles (system poles) of a linear system $\dot{x} = Ax$ are defined by the roots of the characteristic polynomial $\phi(s) := \det(sI - A)$. In addition, a stable pole, an unstable pole, and a neutral pole are defined as poles lie in the open left half-plane, open right half-plane, and the imaginary axis of the complex plane, respectively.

We define the sets $\mathbb{C}_0 := \{s \,|\, \mathrm{Re}[s] = 0\}$ and $\mathbb{C}_+ := \{s \,|\, \mathrm{Re}[s] \geq 0\}$. For a system $\mathcal{S}$, the transfer function representation is denoted by $\tilde{\mathcal{S}}(s)$. A linear time-invariant system $\mathcal{S}$ is said to be positive real [12] (strictly positive real) or PR (SPR) for short if the transfer function $\tilde{\mathcal{S}}(s)$

This research is supported by the Aihara Innovative Mathematical Modelling Project, the Japan Society for the Promotion of Science (JSPS) through the Funding Program for World-Leading Innovative R&D on Science and Technology (FIRST Program), initiated by the Council for Science and Technology Policy (CSTP) and the Grant-in-Aid for Young Scientists (B), No.26820163 and Grant-in-Aid for Challenging Exploratory Research, No.26630195 from JSPS.

[1] M. Inoue is with Japan Science and Technology Agency and Tokyo Institute of Technology, 2-12-1-W8-1, O-okayama, Meguro-ku, Tokyo 152-8552, Japan. `minoue@cyb.mei.titech.ac.jp`

[2] J.-I. Imura and T. Ishizaki are with Graduate School of Information Science and Engineering, Tokyo Institute of Technology. 2-12-1-W8-1, O-okayama, Meguro-ku, Tokyo 152-8552, Japan.

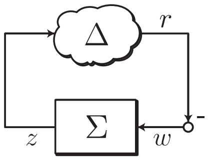

Fig. 2. Feedback system.

is in $\mathcal{RH}_\infty$ and $\bar{\mathcal{S}}(s) + \bar{\mathcal{S}}^*(s) \geq 0$ ($\bar{\mathcal{S}}(s) + \bar{\mathcal{S}}^*(s) \geq \varepsilon I$, where $\varepsilon > 0$) holds for all $s \in \mathbb{C}_+$. In some literature (e.g., [13]), the SPR system in the above sense is said to be strong SPR. Since they are equivalent for a system that satisfies $\bar{\mathcal{S}}(\infty) + \bar{\mathcal{S}}^*(\infty) > 0$, we adopt the term of SPR.

II. INSTABILITY ANALYSIS

A. System Description

We consider the feedback system of Fig. 2. In the figure, the symbol Σ represents the finite-dimensional linear system described by the transfer function matrix

$$z(s) = \bar{\Sigma}(s)w(s),$$

where $w \in \mathbb{R}^m$ and $z \in \mathbb{R}^m$ are the input and output of Σ, respectively. The signal w is given by $w = -r$, where $r \in \mathbb{R}^m$. The symbol Δ represents any dynamic uncertainty that belong to the set of positive-real linear systems such that $r(s) = \bar{\Delta}(s)z(s)$ holds and $\Delta \in \mathcal{U}_{\rm PR}$, where

$$\mathcal{U}_{\rm PR} := \{\, \mathcal{S} \,|\, \bar{\mathcal{S}}(s) \in \mathcal{RH}_\infty,\ \bar{\mathcal{S}}(s) + \bar{\mathcal{S}}^*(s) \geq 0, \forall s \in \mathbb{C}_+ \,\}.$$

This set $\mathcal{U}_{\rm PR}$ includes any uncertainty with high input-output gain as long as its phase difference between w and z is less than or equal to $\pi/2$.

For exposition simplicity, we set an assumption on Σ.

A1) Σ has no neutral poles

In this paper, we derive instability conditions for the feedback system composed of Σ and $\Delta \in \mathcal{U}_{\rm PR}$. To this end, we define *generalized strictly positive realness* even for unstable systems.

Definition 1: *(generalized strictly positive realness, GSPR)* A linear time-invariant system Σ is said to be generalized strictly positive real or GSPR for short if it is hyperbolic and

$$\bar{\Sigma}(s) + \bar{\Sigma}^*(s) \geq \varepsilon I > 0, \quad \forall s \in \mathbb{C}_0 \tag{1}$$

holds.

This definition of generalized *strictly* positive realness is straightforward specialized from that of generalized positive realness (GPR), which is defined for a hyperbolic system $\mathcal{S}$ satisfying (1) for $\varepsilon = 0$ in [12], [14]. However, we remark that the connections in the SPR and GSPR properties are not completely the same as that in the PR and SPR properties. Let us consider a linear time-invariant system $\mathcal{S}$ satisfying $\bar{\mathcal{S}}(\infty) + \bar{\mathcal{S}}^*(\infty) > 0$. On the one hand, $\mathcal{S}$ is SPR if and only if the transfer function $\bar{\mathcal{S}}(s - \delta)$ satisfies the PR property for some positive constant δ (a system having the latter property is said to be SPR in some literature, e.g, [13], [15]). On the other hand, the existence of δ such that $\bar{\mathcal{S}}(s - \delta)$ is GPR does not imply that $\mathcal{S}$ is GSPR if $\mathcal{S}$ has at least one unstable pole. For example, we consider the transfer function $\bar{\mathcal{S}}(s) = 1/(s-1) + 1/2$, which is hyperbolic and unstable but is not GSPR. However, $\bar{\mathcal{S}}(s - \delta)$ becomes GPR for a $\delta = 1$.

By using the GSPR property, we can derive a condition for instability of the feedback system composed of Σ and the positive real-type uncertainty Δ.

B. Main Result: Instability of Feedback System

First, we consider the feedback system composed of Σ and $\Delta \in \mathcal{U}_{\rm PR}$ to derive an instability condition.

Theorem 1: Assume that A1 holds. Then, the feedback system composed of Σ and Δ is well-posed, hyperbolic, and has the same number of the unstable poles as Σ has for all $\Delta \in \mathcal{U}_{\rm PR}$ if Σ is GSPR.

Proof: We prove the theorem by using the Nyquist stability criterion [16], [17]. To prove the theorem by contradiction, we assume that $\det\{I_m + \bar{\Sigma}(j\omega)\bar{\Delta}(j\omega)\}$ intersects or contacts the negative real axis at an $\omega = \omega_0 \in \mathbb{R} \cup \{\pm\infty\}$, i.e., the inequality

$$\det\{I_m + \bar{\Sigma}(j\omega_0)\bar{\Delta}(j\omega_0)\} \leq 0 \tag{2}$$

holds. Then, there exists a positive constant ρ such that $\det\{I_m + \rho\bar{\Sigma}(j\omega_0)\bar{\Delta}(j\omega_0)\} = 0$ holds. The input-output system $\Delta_\rho := \rho\Delta$ is still in $\mathcal{U}_{\rm PR}$. Since this implies that the matrix $\bar{\Sigma}(j\omega_0)\bar{\Delta}_\rho(j\omega_0)$ has the eigenvalue -1, there exists a non-zero vector $v \in \mathbb{C}^m$ such that

$$\bar{\Sigma}(j\omega_0)\bar{\Delta}_\rho(j\omega_0)v = -v$$

holds. We multiply the both sides of the equation by $v^*\bar{\Delta}_\rho^*(j\omega_0)(\neq 0)$ to obtain

$$v^*\bar{\Delta}_\rho^*(j\omega_0)\bar{\Sigma}(j\omega_0)\bar{\Delta}_\rho(j\omega_0)v = -v^*\bar{\Delta}_\rho^*(j\omega_0)v.$$

Then,

$$\begin{aligned} & v^*\bar{\Delta}_\rho^*(j\omega_0)\{\bar{\Sigma}(j\omega_0) + \bar{\Sigma}^*(j\omega_0)\}\bar{\Delta}_\rho(j\omega_0)v \\ &= v^*\bar{\Delta}_\rho^*(j\omega_0)\bar{\Sigma}(j\omega_0)\bar{\Delta}_\rho(j\omega_0)v \\ &\quad + \{v^*\bar{\Delta}_\rho^*(j\omega_0)\bar{\Sigma}(j\omega_0)\bar{\Delta}_\rho(j\omega_0)v\}^* \\ &= -v^*\{\bar{\Delta}_\rho(j\omega_0) + \bar{\Delta}_\rho^*(j\omega_0)\}v \end{aligned}$$

holds. Since Δ_ρ is PR, it follows that

$$v^*\bar{\Delta}_\rho^*(j\omega_0)\{\bar{\Sigma}(j\omega_0) + \bar{\Sigma}^*(j\omega_0)\}\bar{\Delta}_\rho(j\omega_0)v \leq 0. \tag{3}$$

However, we note that from the GSPR property of Σ

$$v_s^*\{\bar{\Sigma}(j\omega) + \bar{\Sigma}^*(j\omega)\}v_s \geq \varepsilon v_s^* v_s \tag{4}$$

holds for all $\omega \in \mathbb{R} \cup \{\pm\infty\}$ and non-zero vectors $v_s \in \mathbb{C}^m$. Therefore, (3) contradicts (4) for a $v_s = \bar{\Delta}_\rho(j\omega_0)v$ and an $\omega = \omega_0$. Then, $\det\{I_m + \bar{\Sigma}(j\omega)\bar{\Delta}_\rho(j\omega)\}$ does not contact the negative real axis.

Since $\det\{I_m + \bar{\Sigma}(\infty)\bar{\Delta}_\rho(\infty)\} \neq 0$, the feedback system composed of Σ and Δ_ρ is well-posed. In addition, we can show that $\det\{I_m + \bar{\Sigma}(j\omega)\bar{\Delta}_\rho(j\omega)\}$ does not encircle the origin. From the Nyquist stability criterion, the feedback system is hyperbolic and has the same number of the unstable poles as Σ has for all $\Delta_\rho \in \mathcal{U}_{\mathrm{PR}}$. ■

Remark 1: For evaluating the PR or SPR property of a system, it is well-known that the Positive Real Lemma [12] or more generally the Kalman-Yakubovich-Popov Lemma [18] provides algebraic conditions based on the system matrices as well as the equivalence between such frequency domain properties and the time-domain algebraic conditions. Such algebraic conditions are computationally tractable to determine if a multi-variable linear time-invariant system is PR (SPR) or not. In a similar way to the lemmas, we propose an algebraic condition for the GSPR property as follows.

The system Σ is described by the state equation

$$\Sigma : \begin{cases} \dot{x} \;=\; Ax + Bw, \\ z \;=\; Cx + Dw, \end{cases} \tag{5}$$

where $x \in \mathbb{R}^n$ is the state and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$ are the constant matrices. From this expression, the transfer function matrix $\bar{\Sigma}(s)$ is given by

$$\bar{\Sigma}(s) = C(sI_n - A)^{-1}B + D.$$

In this state-space realization, the assumption A1 implies that the origin $x = 0$ is hyperbolic. In addition, we set an assumption on the realization of Σ.

A2) The realization (5) is controllable and observable.

Then, we have the following algebraic condition for the GSPR property.

Proposition 1: Assume that A1 and A2 hold. Then, Σ is GSPR if and only if there exist matrices P, $Q \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times m}$, and $W \in \mathbb{R}^{m\times m}$ such that P is symmetric, Q is positive definite, and

$$\begin{aligned} PA + A^{\mathsf{T}}P \;&=\; -LL^{\mathsf{T}} - Q \\ PB - C^{\mathsf{T}} \;&=\; -LW \\ D + D^{\mathsf{T}} \;&=\; W^{\mathsf{T}}W \end{aligned}$$

hold.

This proposition is derived by decomposing Σ into a stable system and an anti-stable system and using the Positive Real Lemma (see e.g., [13]). The proof is omitted in this paper. Compared with the Positive Real Lemma for stable systems, in the condition of Proposition 1 the positive definiteness of the matrix P is not required. If the algebraic equations applied to an unstable system $\mathcal{S}$ with the GSPR property are solved, we obtain a P that is symmetric and nonsingular but is not positive definite. The matrix P has just the same number of the negative eigenvalues as that of the unstable poles in Σ. We can show this fact by using the inertia theorem [19].

III. FURTHER CONSIDERATION AND FUTURE PERSPECTIVES

A. Numerical Example: Robustness Analysis of Genetic Oscillator Model

In this subsection, we consider an application of Theorem 1 to analysis of a nonlinear system. We analyze the robustness of the genetic oscillator model [1] with dynamic uncertainties to illustrate the validity as well as the limitation of the theorem for linear systems.

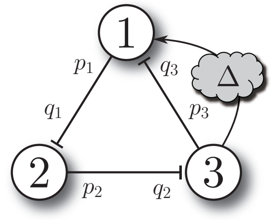

Fig. 3. Genetic network model with uncertainty

The genetic oscillator is illustrated in Fig. 3, where each node is described by the nonlinear differential equations

$$\Sigma : \begin{cases} \dot{m}_i \;=\; -m_i + g(q_j), \quad g(q) = \dfrac{\alpha}{1+q^2}, \\ \dot{p}_i \;=\; -\beta(p_i - m_i), \\ \qquad\qquad (i,j) = (1,3), (2,1), (3,2), \end{cases}$$

where m_i, $i = \{1,2,3\}$ are the messenger RNA (mRNA) concentrations, p_i, $i = \{1,2,3\}$ are the repressor-protein concentrations, q_j, $j = \{1,2,3\}$ are the inputs, and α, $\beta > 0$ denote the protein copies per a cell when repressor is absent, the ratio of the protein decay rate to the mRNA decay rate, respectively. The symbol Δ in the figure expresses the dynamic uncertainty such as affections from other gene expressions in living cells, which are ignored in the modeling process. Then, the links in the figure are described by $q_1 = p_1$, $q_2 = p_2$, and

$$q_3(s) = (1 - \bar{\Delta}(s))p_3(s),$$

where $\bar{\Delta}(s)$ is the transfer function of Δ, which belongs to a set of uncertainties $\mathcal{U}$. If the model has no uncertainty and $q_j = p_j$, $j = \{1,2,3\}$ hold, a negative feedback loop is constructed. As illustrated in Fig. 3, p_1 inhibits m_2 and p_2, and p_2 inhibits m_3 and p_3, and p_3 inhibits m_1 and p_1.

The behavior of the system of Fig. 3 is dependent on the parameter values (α, β) and the dynamic uncertainty $\Delta \in \mathcal{U}$. When $\Delta = 0$, two types of behavior are possible and we can find them by stability and instability analysis of the unique equilibrium. The behavior of $m_i(t)$ and $p_i(t)$ converges to fixed values when the equilibrium is asymptotically stable. Conversely, sustainable oscillations appear if the equilibrium is unstable [1], [7]. For example, for the parameter values of $(\alpha, \beta) = (20, 1)$, we can see the oscillatory behavior in $m_i(t)$, $i = \{1,2,3\}$ as illustrated in Fig. 4. Even in the uncertain case, we can roughly know the behavior by stability and instability analysis of the equilibrium [9].

Now, let us transform the network model of Fig. 3 to the feedback form of Fig. 2. First, we find the unique equilibrium point $p_i(t) \equiv p_e$ and $m_i(t) \equiv m_e$, $i = \{1,2,3\}$ and define new variables $\tilde{p}_i = p_i - p^e$, $\tilde{m}_i = m_i - m^e$, $i = \{1,2,3\}$, and w and $z = \tilde{p}_3$. Then, we obtain the negative feedback

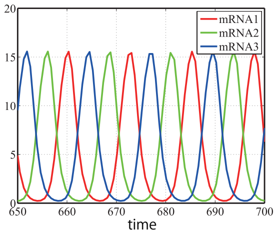

Fig. 4. Oscillatory behavior for $(\alpha, \beta) = (20, 1)$

system composed of the error dynamics

$$z(s) = \bar{\Sigma}_e(s) w(s),$$

$$\Sigma_e : \begin{cases} \dot{\tilde{m}}_i &= -\tilde{m}_i + f(\tilde{q}_j),\ f(\tilde{q}_j) = g(\tilde{q}_j + p^e) - g(p^e), \\ \dot{\tilde{p}}_i &= -\beta(\tilde{p}_i - \tilde{m}_i), \\ \tilde{q}_1 &= \tilde{p}_1, \quad \tilde{q}_2 = \tilde{p}_2, \quad \tilde{q}_3 = \tilde{p}_3 + w, \\ & \qquad\qquad (i, j) = (1, 3), (2, 1), (3, 2) \end{cases}$$

and $w = -r$, and the dynamic uncertainty

$$r(s) = \bar{\Delta}(s) z(s).$$

We assume that the parameter values are given by $(\alpha, \beta) = (20, 1)$. Then, the origin of Σ_e is hyperbolic and its linearized system has two unstable poles. In addition, Σ_e is not GSPR, but $\Sigma_e + \epsilon$ can be GSPR for some $\epsilon \geq \epsilon_0 = 1.43 \times 10^{-4}$. From Theorem 1, the origin of the linear feedback system composed of the linearized Σ_e and the uncertainty $\Delta \in \mathcal{U}_{\mathrm{PR}+}(\epsilon_0)$,

$$\mathcal{U}_{\mathrm{PR}+}(\epsilon_0) := \left\{ \mathcal{S} \,\middle|\, \bar{\mathcal{S}}(s) = \frac{\bar{\mathcal{D}}(s)}{1 + \epsilon \bar{\mathcal{D}}(s)},\ \mathcal{D} \in \mathcal{U}_{\mathrm{PR}},\ \forall \epsilon \geq \epsilon_0 \right\}$$

is hyperbolic and has two unstable poles as well. We can expect that if Δ is in some class $\mathcal{U} \subseteq \mathcal{U}_{\mathrm{PR}+}(\epsilon_0)$, even the uncertain genetic network model of Fig. 3, i.e., the nonlinear feedback system composed of the nonlinear Σ_e and Δ, has a hyperbolic equilibrium and its linearized system has two unstable poles. In addition, the existence of a stable limit cycle is expected although it is not proved theoretically.

First, consider the case that $\bar{\Delta}(0) = 0$ holds, i.e., the steady state gain of the uncertainty Δ is 0. Then, the location of the unique equilibrium of the nominal Σ is not shifted by the uncertainty $\Delta \in \{\Delta \,|\, \bar{\Delta}(0) = 0\}$. In other words, the origin is the equilibrium of the nonlinear feedback system composed of Σ_e and Δ for all $\Delta \in \{\Delta \,|\, \bar{\Delta}(0) = 0\}$. The analysis result based on the linearized Σ_e is also valid for the nonlinear Σ_e. The origin of the nonlinear feedback system composed of Σ_e and Δ is hyperbolic and its linearized system has the same number of unstable poles as the linearized Σ_e if $\Delta \in \mathcal{U}_{\mathrm{PR}+}(\epsilon_0) \cap \{\Delta \,|\, \bar{\Delta}(0) = 0\}$. We expect that the oscillation appears in the feedback system if $\Delta \in \mathcal{U}_{\mathrm{PR}+}(\epsilon_0) \cap \{\Delta \,|\, \bar{\Delta}(0) = 0\}$. We simulate the behavior of the genetic system of Fig. 3 with a fixed $\Delta = \Delta_{\mathrm{ex}}$ of

$$\bar{\Delta}_{\mathrm{ex}}(s) = \frac{1000s + \delta_0}{2000s + 1},$$

where δ_0 is the non-negative constant and $\Delta_{\mathrm{ex}} \in \mathcal{U}_{\mathrm{PR}+}(\epsilon_0)$. As illustrated in Fig. 5, we have the oscillatory behavior for $\delta_0 = 0$.

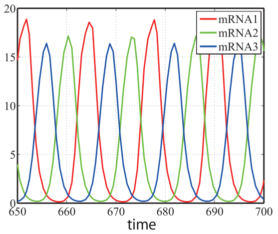

Fig. 5. Behavior of the feedback system for $\delta_0 = 0$

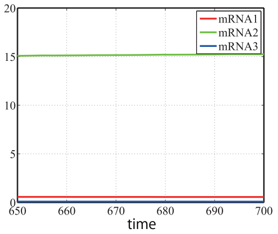

Fig. 6. Behavior of the feedback system for $\delta_0 = 100$

Next, let us consider the case that $\delta_0 = 100$ holds in Δ_{ex}. Then, the origin is not the equilibrium of the nonlinear feedback system composed of Σ_e and Δ_{ex}. In addition, since the linearized $\Sigma_e + \epsilon$, $\epsilon \geq \epsilon_0$ at the equilibrium whose location is shifted by Δ_{ex} is not GSPR. Then, the equilibrium of the feedback system becomes stable. As illustrated in Fig. 6, the nonlinear feedback system loses the oscillatory behavior. This example shows that the instability and GSPR property analysis based on the linearized subsystem is not valid for analysis of a nonlinear system. We need to consider that the equilibrium of a nonlinear system depends on uncertainties. In future works, we will derive a new instability condition for systems with uncertainty-dependent equilibrium by using similar concepts of the parametric stability [20], [21], [22] or the equilibrium-independent passivity [23], which can consider the existence of the uncertainty-dependent equilibrium for the purpose of stability or passivity (positive realness) analysis.

B. *Discussion on Instability of Positive Real Network*

We remark future extensions of the instability analysis based on Theorem 1. In this section, we consider the network system of Fig. 7 (a).

In the figure, the dynamics in each node is represented by the state equation

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + B_i u_i \\ y_i = C_i x_i + D_i u_i \end{cases}, \quad i \in \{1, 2, \ldots, N\}$$

where $x_i \in \mathbb{C}^{n_i}$, u_i, and $y_i \in \mathbb{C}^{m_i}$ are the state, input, and output of the i-th node, and $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $C_i \in \mathbb{R}^{m_i \times n_i}$, and $D_i \in \mathbb{R}^{m_i \times m_i}$ are the constant coefficient matrices. Each node Σ_i is connected with others by the rule

$$u_i = -\sum_{j=1}^{N} L_{ij} y_j, \quad i \in \{1, 2, \ldots, N\}, \tag{6}$$

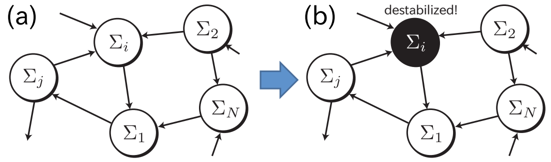

Fig. 7. Network model with destabilized node

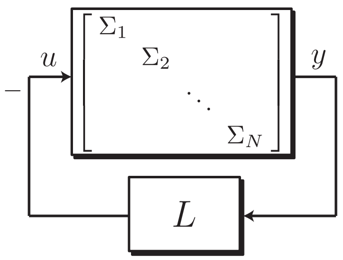

Fig. 8. Feedback representation of network system

where $L_{ij} \in \mathbb{R}^{m_i \times m_i}$, $i, j \in \{1, 2, \ldots, N\}$ are the constant matrix. Although this rule includes the error feedback connection $u_i = \sum_{i \neq j} k_{ij}(x_i - x_j)$ for example, no special structure is considered. In the following discussion, the network connection is denoted by

$$u = -Ly, \quad L := \{L_{ij}\} \in \mathbb{R}^{M \times M}, \quad M = \sum_{i}^{N} m_i,$$

where the signals u and $y \in \mathbb{R}^M$ are defined by $u := [\,u_1^\mathsf{T}\, u_2^\mathsf{T} \cdots u_N^\mathsf{T}\,]^\mathsf{T}$ and $y := [\,y_1^\mathsf{T}\, y_2^\mathsf{T} \cdots y_N^\mathsf{T}\,]^\mathsf{T}$. Then, the network system is represented by the feedback form of Fig. 8.

We review a positive real network system, i.e., a network system composed of positive real subsystems which are internally connected in a specific network topology. As implied by the passivity-stability theorem, the origin of the network system is stable if Σ_i, $i \in \{1, 2, \ldots, N\}$ are SPR and $L + L^\mathsf{T}$ is positive semi-definite. In a similar way, we can derive the following instability condition for the network system from Theorem 1.

Corollary 1: Assume that Σ_i, $i \in \{1, 2, \ldots, N\}$ are hyperbolic and have $P_i \geq 0$, $i \in \{1, 2, \ldots, N\}$ unstable poles, respectively. Then, the origin of the network system composed of Σ_i, $i \in \{1, 2, \ldots, N\}$ by the rule (6) is well-posed, hyperbolic, and has $P = \sum_i^N P_i$ unstable poles if Σ_i, $i \in \{1, 2, \ldots, N\}$ are GSPR and $L + L^\mathsf{T}$ is positive semi-definite.

It is known that the positive real network construction is a typical strategy that the stability of the entire network is guaranteed only by properties in local nodes and a connection rule such as negative feedback connection and additive connection. From Corollary 1, we can see that the positive real network has a disadvantage that it becomes unstable if at least one of the nodes is destabilized with preserving the GSPR property as illustrated in Fig. 7 (a) and (b), while non-positive real network may preserve stability in the same situation. We need further consideration to conclude that a positive real network is good or bad.

IV. CONCLUSION

In this paper, we defined the generalized strictly positive realness for hyperbolic linear systems. Based on the property, we derived a condition for instability of a feedback system composed of a nominal linear system and a positive real uncertainty. Finally, we remarked for future perspectives of instability analysis of uncertain systems and network systems.

ACKNOWLEDGEMENTS

The authors acknowledge Takayuki Arai and Le Thi Dieu Hang for their helps and comments on the numerical analysis in Section III-A.

REFERENCES

[1] M. B. Elowitz and S. Leibler, "A synthetic oscillatory network of transcriptional regulators," *Nature*, vol. 403, no. 6767, pp. 335–338, 2000.

[2] L. Ma and P. A. Iglesias, "Quantifying robustness of biochemical network models," *BMC Bioinformatics*, vol. 3, no. 1, p. 38, 2002.

[3] M. R. Atkinson, M. A. Savageau, J. T. Myers, and A. J. Ninfa, "Development of genetic circuitry exhibiting toggle switch or oscillatory behavior in Escherichia coli," *Cell*, vol. 113, no. 5, pp. 597–607, 2003.

[4] E. Fung, W. W. Wong, J. K. Suen, T. Bulter, S.-g. Lee, and J. C. Liao, "A synthetic gene–metabolic oscillator," *Nature*, vol. 435, no. 7038, pp. 118–122, 2005.

[5] W. W. Wong, T. Y. Tsai, and J. C. Liao, "Single-cell zeroth-order protein degradation enhances the robustness of synthetic oscillator," *Molecular Systems Biology*, vol. 3, no. 1, 2007.

[6] C. Trané and E. W. Jacobsen, "Network structure and robustness of intracellular oscillators," in *Proceedings of the 17th IFAC World Congress*, 2008, pp. 10 989–10 994.

[7] Y. Hori, T.-H. Kim, and S. Hara, "Existence criteria of periodic oscillations in cyclic gene regulatory networks," *Automatica*, vol. 47, no. 6, pp. 1203–1209, 2011.

[8] S. Waldherr and F. Allgöwer, "Robust stability and instability of biochemical networks with parametric uncertainty," *Automatica*, vol. 47, no. 6, pp. 1139–1146, 2011.

[9] M. Inoue, J.-I. Imura, K. Kashima, T. Arai, and K. Aihara, "An instability condition for uncertain systems toward robust bifurcation analysis," in *Proceedings of the European Control Conference 2013*, 2013, pp. 3264–3269.

[10] M. Inoue, J.-I. Imura, K. Kashima, and K. Aihara, "Robust bifurcation analysis based on the Nyquist stability criterion," in *Proceedings of the 52nd IEEE Conference on Decision and Control*, 2013, pp. 1768–1773.

[11] ——, "Robust bifurcation analysis of systems with dynamic uncertainties," *International Journal of Bifurcation and Chaos*, vol. 23, no. 9, p. 1350157, 2013.

[12] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*. Prentice Hall, 1973.

[13] B. Brogliato, R. Lozano, B. Maschke, O. Egeland *et al.*, *Dissipative Systems Analysis and Control: Theory and Applications*. Springer, 2007.

[14] B. D. O. Anderson and J. B. Moore, "Algebraic structure of generalized positive real matrices," *SIAM Journal on Control*, vol. 6, no. 4, pp. 615–624, 1968.

[15] K. S. Narendra and J. H. Taylor, *Frequency Domain Criteria for Absolute Stability*. Access Online via Elsevier, 1973.

[16] K. Zhou, J. C. Doyle, K. Glover *et al.*, *Robust and Optimal Control*. Prentice Hall, 1996.

[17] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control: Analysis and Design*, 2nd ed. Wiley-Interscience, 2005.

[18] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma," *Systems & Control Letters*, vol. 28, no. 1, pp. 7–10, 1996.

[19] A. C. Antoulas, *Approximation of Large-Scale Dynamical Systems*. SIAM, 2005.

[20] M. Ikeda, Y. Ohta, and D. D. Šiljak, “Parametric stability,” in *New Trends in Systems Theory*. Springer, 1991, pp. 1–20.

[21] T. Wada, M. Ikeda, Y. Ohta, and D. D. Šiljak, “Parametric absolute stability of Lur’e systems,” *Automatic Control, IEEE Transactions on*, vol. 43, no. 11, pp. 1649–1653, 1998.

[22] ——, “Parametric absolute stability of multivariable Lur’e systems,” *Automatica*, vol. 36, no. 9, pp. 1365–1372, 2000.

[23] G. H. Hines, M. Arcak, and A. K. Packard, “Equilibrium-independent passivity: A new definition and numerical certification,” *Automatica*, vol. 47, no. 9, pp. 1949–1956, 2011.