

Conditions for Dissipativeness of 2-D Discrete-Time Behaviors Based on Quadratic Difference Forms

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Abstract—In this paper, we consider losslessness and dissipativeness for two-dimensional (2-D) behaviors using quadratic difference forms (QDFs) based on the behavioral approach. We derive necessary and sufficient conditions for the losslessness and dissipativeness in terms of dissipation equality and a certain frequency domain inequality as a main result.

I. INTRODUCTION

Dissipativeness is one of the most important properties of a dynamical system which captures the system from a view point of energy interactions with its external environment [5]. Willems and Trentelman [7] proved that a dissipativeness of a system is equivalent to a certain frequency domain inequality and the dissipation inequality in terms of quadratic differential forms (QDFs), which are useful algebraic tools for the dissipation theory in the behavioral approach [6], for one-dimensional (1-D) systems. The result of [7] was extended to n -dimensional (n -D) systems¹ by [3].

In the discrete-time case, Kaneko and Fujii [1] developed the dissipation theory for 1-D systems based on QDFs, where the term QDF stands for *quadratic difference form* for discrete-time systems. However, for discrete-time n -D systems, there have never been derived a necessary and sufficient condition for dissipativeness in terms of dissipation and frequency domain inequalities despite of its importance in iterative learning control, digital image processing and etc. Hence, we have conceived to derive conditions equivalent to dissipativeness for 2-D discrete-time systems. As it was pointed out that there exists a difficulty in a spectral factorization in n -D systems [3], it is not straightforward to generalize the previous results [3][1] for the 2-D discrete-time case. We show how this difficulty can be resolved in this case.

We use the following notations in this paper. The sets of $p \times q$ real and $q \times q$ real symmetric matrices are denoted by $\mathbb{R}^{p \times q}$ and $\mathbb{R}_s^{q \times q}$, respectively. We denote the indeterminates $\xi := (\xi_1, \xi_2)$ and $\zeta := (\zeta_1, \zeta_2)$, $\eta := (\eta_1, \eta_2)$, when we consider two- and four-variable polynomial matrices, respectively. The sets of $p \times q$ real coefficient two- and four-variable polynomial matrices are denoted by $\mathbb{R}^{p \times q}[\xi]$ and

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¹We call a system that depends on n independent variables ($n \geq 2$) as an n -dimensional system

$\mathbb{R}^{p \times q}[\zeta, \eta]$, respectively. We also denote $\mathbb{R}_s^{q \times q}[\zeta, \eta]$ as the set of $q \times q$ symmetric four-variable polynomial matrices, i.e. $\Phi(\zeta, \eta)^\top = \Phi(\eta, \zeta)$ for any $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. We denote the matrix $[A_1^\top \ A_2^\top \ \cdots \ A_n^\top]^\top$ by $\text{col}(A_1, A_2, \dots, A_n)$. The symbol $\mathbb{W}^\mathbb{T}$ denotes the set of maps from \mathbb{T} to \mathbb{W} . Finally, we define the set

$$l_2^q := \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \mid \sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} \|w(t_1, t_2)\|^2 < \infty \right\}.$$

II. PRELIMINARIES

A. 2-D Discrete-Time System

In the behavioral system theory [6], a dynamical system is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where \mathbb{T} is the set of independent variables, and \mathbb{W} is the signal space in which the trajectories take their values on. The behavior $\mathfrak{B} \subseteq \mathbb{W}^\mathbb{T}$ is the set of all possible trajectories. In this paper, we consider a *two-dimensional (2-D) linear time-invariant discrete-time system* $\Sigma = (\mathbb{Z}^2, \mathbb{R}^q, \mathfrak{B})$ which is represented by the linear partial difference-algebraic equation

$$R(\sigma_1, \sigma_2)w = 0, \quad (1)$$

where $R(\xi) := \sum_{i_1=0}^{L_1} \sum_{i_2=0}^{L_2} R_{i_1, i_2} \xi_1^{i_1} \xi_2^{i_2} \in \mathbb{R}^{p \times q}[\xi]$, $R_{i_1, i_2} \in \mathbb{R}^{p \times q}$ ($i_k = 0, 1, \dots, L_k$; $k = 1, 2$) and $L_k \geq 0$. The operator σ_k ($k = 1, 2$) is the shift operator on $t_k \in \mathbb{Z}$ defined by $(\sigma_1 w)(t_1, t_2) := w(t_1 + 1, t_2)$, $(\sigma_2 w)(t_1, t_2) := w(t_1, t_2 + 1)$ and $(\sigma_1^{T_1} \sigma_2^{T_2} w)(t_1, t_2) := w(t_1 + T_1, t_2 + T_2)$, $T_1, T_2 \in \mathbb{Z}$. Then, the behavior is the kernel of the shift operator $R(\sigma_1, \sigma_2)$ given by

$$\mathfrak{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \mid R(\sigma_1, \sigma_2)w = 0 \right\}. \quad (2)$$

For this reason, (1) is called the *kernel representation* of \mathfrak{B} .

The behavior \mathfrak{B} is called *controllable* if there exists a $\rho > 0$ such that for any $w_1, w_2 \in \mathfrak{B}$ and $T_1, T_2 \subset \mathbb{Z}^2$ with $d(T_1, T_2) > \rho$ there exists a $w \in \mathfrak{B}$ such that $w(t_1, t_2) = w_1(t_1, t_2)$ ($(t_1, t_2) \in T_1$) and $w(t_1, t_2) = w_2(t_1, t_2)$ ($(t_1, t_2) \in T_2$), where $d(T_1, T_2) := \min \left\{ \sum_{k=1}^2 |t_{1,k} - t_{2,k}| \mid (t_{1,1}, t_{1,2}) \in T_1, (t_{2,1}, t_{2,2}) \in T_2 \right\}$ [8].

The behavior \mathfrak{B} is controllable if and only if it can be described by an *image representation*

$$w = M(\sigma_1, \sigma_2)\ell, \quad (3)$$

where $M \in \mathbb{R}^{q \times m}[\xi]$ and $\ell \in (\mathbb{R}^m)^{\mathbb{Z}^2}$ is called the *latent variable* [8]. Then, \mathfrak{B} is given by $\mathfrak{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \mid \exists \ell \in (\mathbb{R}^m)^{\mathbb{Z}^2} \text{ s.t. (3)} \right\}$.

When \mathfrak{B} is represented by an image representation, \mathfrak{B} is called *observable* if $w = M(\sigma_1, \sigma_2)\ell = 0$ implies $\ell = 0$. The representation (3) is observable if and only if the constant matrix $M(\lambda)$ is of full column rank for all $\lambda \in (\mathbb{C} \setminus \{0\})^2$ [4]. For 2-D behaviors, every controllable behavior does not necessarily have an observable image representation contrary to the 1-D case [4].

B. Quadratic Difference Forms

The *quadratic difference form (QDF)* $Q_\Phi(\ell)$ is a quadratic form of the variables $\ell \in (\mathbb{R}^q)^{\mathbb{Z}^2}$ and its shifts, namely

$$Q_\Phi : (\mathbb{R}^q)^{\mathbb{Z}^2} \rightarrow \mathbb{R}^{\mathbb{Z}^2},$$

$$Q_\Phi(\ell) := \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} (\sigma_1^{i_1} \sigma_2^{i_2} \ell)^\top \Phi_{i_1, i_2, j_1, j_2} \sigma_1^{j_1} \sigma_2^{j_2} \ell,$$

where $\Phi_{i_1, i_2, j_1, j_2} = \Phi_{j_1, j_2, i_1, i_2}^\top \in \mathbb{R}^{q \times q}$. There is a one-to-one correspondence between the QDF and the four-variable polynomial matrix

$$\Phi(\zeta, \eta) = \sum_{i_1=0}^{N_1} \sum_{i_2=0}^{N_2} \sum_{j_1=0}^{N_1} \sum_{j_2=0}^{N_2} \Phi_{i_1, i_2, j_1, j_2} \zeta_1^{i_1} \zeta_2^{i_2} \eta_1^{j_1} \eta_2^{j_2}.$$

This means that (ζ_1, ζ_2) and (η_1, η_2) correspond to the shift operations on $\ell(t_1, t_2)^\top$ and $\ell(t_1, t_2)$, respectively.

A four-variable polynomial matrix

$$\Psi(\zeta, \eta) = \begin{bmatrix} \Psi_1(\zeta, \eta) \\ \Psi_2(\zeta, \eta) \end{bmatrix}, \quad \Psi_1, \Psi_2 \in \mathbb{R}_s^{q \times q}[\zeta, \eta] \quad (4)$$

induces a *vector of QDFs*

$$Q_\Psi : (\mathbb{R}^q)^{\mathbb{Z}^2} \rightarrow (\mathbb{R}^2)^{\mathbb{Z}^2}, \quad Q_\Psi(\ell)(t_1, t_2) = \begin{bmatrix} Q_{\Psi_1}(\ell)(t_1, t_2) \\ Q_{\Psi_2}(\ell)(t_1, t_2) \end{bmatrix}.$$

The *discrete-time divergence* of the vector of QDF $Q_\Psi(\ell)$ is defined by

$$\nabla Q_\Psi(\ell)(t_1, t_2) := \{Q_{\Psi_1}(\ell)(t_1 + 1, t_2) - Q_{\Psi_1}(\ell)(t_1, t_2)\} \\ + \{Q_{\Psi_2}(\ell)(t_1, t_2 + 1) - Q_{\Psi_2}(\ell)(t_1, t_2)\}.$$

This is also a QDF. Let $\nabla \Psi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ induce $\nabla Q_\Psi(\ell)$. Then, it is given by $\nabla \Psi(\zeta, \eta) = \sum_{k=1}^2 (\zeta_k \eta_k - 1) \Psi_k(\zeta, \eta)$.

III. PATH INDEPENDENCE

This section derives a necessary and sufficient condition for path independence of 2-D discrete-time behaviors.

Consider the summation

$$\sum_{(t_1, t_2) \in \Omega \cap \mathbb{Z}^2} Q_\Phi(w)(t_1, t_2), \quad (5)$$

where Ω is a closed bounded subset of \mathbb{R}^2 with a nonempty interior. The summation in (5) is said to be *independent of the path* w if for any $w_1, w_2 \in (\mathbb{R}^q)^{\mathbb{Z}^2}$ such that $\sigma_1^{k_1} \sigma_2^{k_2} w_1(t_1, t_2) = \sigma_1^{k_1} \sigma_2^{k_2} w_2(t_1, t_2)$, $\forall (t_1, t_2) \in \partial\Omega \cap \mathbb{Z}^2$, $(k_1, k_2) \in \mathbb{N}^2$, there holds

$$\sum_{(t_1, t_2) \in \Omega \cap \mathbb{Z}^2} Q_\Phi(w_1)(t_1, t_2) = \sum_{(t_1, t_2) \in \Omega \cap \mathbb{Z}^2} Q_\Phi(w_2)(t_1, t_2),$$

where $\partial\Omega$ denotes the boundary of Ω .

Proposition 1: Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be given. Then, the following statements are equivalent.

- (i) The summation in (5) is independent of path for the intersection of all closed bounded subsets Ω and \mathbb{Z}^2 .
- (ii) The following equality holds.

$$\sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_\Phi(w)(t_1, t_2) = 0, \quad \forall w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \cap l_2^q \quad (6)$$

- (iii) The equality $\partial\Phi(\xi) = 0$ holds.
- (iv) There exists a $\Psi \in \mathbb{R}^{2q \times q}[\zeta, \eta]$ in (4) satisfying $\Phi(\zeta, \eta) = \nabla \Psi(\zeta, \eta)$.
- (v) There exists a $\Psi \in \mathbb{R}^{2q \times q}[\zeta, \eta]$ in (4) satisfying

$$Q_\Phi(w)(t_1, t_2) = \nabla Q_\Psi(w)(t_1, t_2) \quad (7)$$

for all $w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \cap l_2^q$ and $(t_1, t_2) \in \mathbb{Z}^2$.

Proof: (i) \Rightarrow (iii) By an analogous discussion to the proof of Theorem 4 (i) \Rightarrow (iii) in [3], the summation in (5) is independent of the path w if and only if

$$\sum_{(t_1, t_2) \in \Omega \cap \mathbb{Z}^2} Q_\Phi(w)(t_1, t_2) + 2 \sum_{(t_1, t_2) \in \Omega \cap \mathbb{Z}^2} L_\Phi(v, w)(t_1, t_2) = 0 \quad (8)$$

holds for any $w \in (\mathbb{R}^q)^{\mathbb{Z}^2} \cap l_2^q$ with support in Ω , $v \in (\mathbb{R}^q)^{\mathbb{Z}^2}$ and closed bounded subset $\Omega \subset \mathbb{R}^2$ with nonempty interior. Now we consider $v = 0$ as a special trajectory. Then, it is necessary that $\sum_{(t_1, t_2) \in \Omega \cap \mathbb{Z}^2} Q_\Phi(w)(t_1, t_2) = 0$ holds. Moreover, we have the following equality from (8).

$$\sum_{(t_1, t_2) \in \Omega \cap \mathbb{Z}^2} L_\Phi(w, v)(t_1, t_2) = 0 \quad (9)$$

Define $\widehat{\Phi}_{k_1, k_2}, \widehat{\Phi}_{k_1, -k_2}, \widehat{\Phi}_{-k_1, k_2}, \widehat{\Phi}_{-k_1, -k_2} \in \mathbb{R}^{q \times q}$ ($k_1 = 0, 1, \dots, N_1, k_2 = 0, 1, \dots, N_2$) by

$$\widehat{\Phi}_{k_1, k_2} := \sum_{i_1=0}^{N_1-k_1} \sum_{i_2=0}^{N_2-k_2} \Phi_{i_1, i_2, i_1+k_1, i_2+k_2},$$

$$\widehat{\Phi}_{k_1, -k_2} := \sum_{i_1=0}^{N_1-k_1} \sum_{i_2=0}^{N_2-k_2} \Phi_{i_1, i_2+k_2, i_1+k_1, i_2},$$

$$\widehat{\Phi}_{-k_1, k_2} := \sum_{i_1=0}^{N_1-k_1} \sum_{i_2=0}^{N_2-k_2} \Phi_{i_1+k_1, i_2, i_1, i_2+k_2},$$

$$\widehat{\Phi}_{-k_1, -k_2} := \sum_{i_1=0}^{N_1-k_1} \sum_{i_2=0}^{N_2-k_2} \Phi_{i_1+k_1, i_2+k_2, i_1, i_2}. \quad (10)$$

From (9) and (10), we can prove

$$\widehat{\Phi}_{k_1, k_2} = 0, \widehat{\Phi}_{k_1, -k_2} = 0, \widehat{\Phi}_{-k_1, k_2} = 0, \widehat{\Phi}_{-k_1, -k_2} = 0 \quad (11)$$

for all $k_1 = 0, 1, \dots, N_1$ and $k_2 = 0, 1, \dots, N_2$ by an induction with respect to k_1 and k_2 . Observe that $\partial\Phi(\xi)$ is expressed as $\partial\Phi(\xi) = \sum_{k_1=-N_1}^{N_1} \xi_1^{k_1} \sum_{k_2=-N_2}^{N_2} \xi_2^{k_2} \widehat{\Phi}_{k_1, k_2}$. Then, the statement (iii) follows from (11).

(iii) \Leftrightarrow (iv) The equivalence of (iii) and (iv) was proved in Proposition 1 of [2].

(iv) \Leftrightarrow (v) The equivalence of (iv) and (v) follows immediately from the definition of a QDF.

(v)⇒(ii) Summing up (7) from $t_1 = -\infty$ to $t_1 = +\infty$ and from $t_2 = -\infty$ and $t_2 = +\infty$, we have (6) from $w \in l_2^q$.
(ii)⇒(i) Assume that (6) holds. Since w is an arbitrary trajectory in l_2^q , we can prove the statement (i). ■

IV. CONDITIONS FOR LOSSLESSNESS AND DISSIPATIVENESS

This section derives necessary and sufficient conditions for losslessness and dissipativeness using QDFs as a main result. Throughout this section, we consider the behavior \mathfrak{B} given by (2), where $w \in (\mathbb{R}^q)^{\mathbb{Z}^2}$ is the manifest variable and $R \in \mathbb{R}^{p \times q}[\zeta]$ induces the kernel representation (1).

A. Lossless and Dissipative Behaviors

We define losslessness and dissipativeness of a behavior.

Let $\Pi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be given. A behavior \mathfrak{B} is called *lossless with respect to the supply rate* $Q_\Pi(w)$ if the equality

$$\sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_\Pi(w)(t_1, t_2) = 0, \quad \forall w \in l_2^q \cap \mathfrak{B}$$

holds. Moreover, \mathfrak{B} is called *dissipative with respect to the supply rate* $Q_\Pi(w)$ if the following inequality holds.

$$\sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_\Pi(w)(t_1, t_2) \geq 0, \quad \forall w \in l_2^q \cap \mathfrak{B}$$

In the remainder of this section, we assume that \mathfrak{B} is controllable. Then, \mathfrak{B} has an image representation (3) which is possibly unobservable. The goal of this paper is to derive a necessary and sufficient condition for the losslessness and dissipativeness in terms of QDFs.

Remark 1: We can think of $Q_\Pi(w)$ as the power delivered to a system. The dissipativeness implies that the net flow of energy into the system is nonnegative, i.e. the system dissipates energy. Hence, the rate of increase of the energy stored inside of the does not exceed the power supplied to it. On the other hand, the losslessness means that the power supplied to the system can be stored as an increase of the internal energy of the system without dissipation.

B. Condition for Losslessness

We give a necessary and sufficient condition for the losslessness in this subsection.

We see from Proposition 1 that \mathfrak{B} is lossless with respect to the supply rate $Q_\Pi(w)$ if and only if the QDF $Q_\Phi(\ell)$ induced by the four-variable polynomial matrix

$$\Phi(\zeta, \eta) = M(\zeta)^\top \Pi(\zeta, \eta) M(\eta) \quad (12)$$

is independent of the path ℓ . Then, we have the following theorem from Proposition 1

Theorem 1: Let the controllable behavior \mathfrak{B} be given by (2). Suppose that \mathfrak{B} is represented by an image representation (3). Define $\Phi \in \mathbb{R}_s^{m \times m}[\zeta, \eta]$ by (12) for a given $\Pi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. Then, the following statements are equivalent.

- (i) The behavior \mathfrak{B} is lossless with respect to the supply rate $Q_\Pi(w)$.

- (ii) The QDF $Q_\Pi(w)$ is independent of path on \mathfrak{B} .
- (iii) The QDF $Q_\Phi(\ell)$ is independent of path ℓ .
- (iv) The equality $\partial\Phi(\xi) = 0$ holds.
- (v) There exists a four-variable polynomial matrix

$$\Psi(\zeta, \eta) = \begin{bmatrix} \Psi_1(\zeta, \eta) \\ \Psi_2(\zeta, \eta) \end{bmatrix}, \quad \Psi_1, \Psi_2 \in \mathbb{R}_s^{m \times m}[\zeta, \eta] \quad (13)$$

satisfying $\nabla Q_\Psi(\ell)(t_1, t_2) = Q_\Pi(w)(t_1, t_2) = Q_\Phi(\ell)(t_1, t_2)$ for all $\ell \in (\mathbb{R}^m)^{\mathbb{Z}^2}$, $(t_1, t_2) \in \mathbb{Z}^2$ with the image representation (3).

Proof: We can prove the equivalence of (i), (ii) and (iii) from $\mathfrak{B} = \ker R(\sigma_1, \sigma_2) = \text{Im}M(\sigma_1, \sigma_2)$ by an analogous discussion to Theorem 7 in [3]. Moreover, the equivalence of (iii), (iv) and (v) follows from Proposition 1 ■

C. Condition for Dissipativeness

We first define the average nonnegativity of a QDF and storage function, dissipation rate.

Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ be given by (12). The QDF $Q_\Phi(\ell)$ is called *average nonnegative* if there holds

$$\sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_\Phi(\ell)(t_1, t_2) \geq 0, \quad \forall \ell \in l_2^m. \quad (14)$$

Then, \mathfrak{B} is dissipative with respect to the supply rate $Q_\Pi(w)$ if and only if the QDF $Q_\Phi(\ell)$ is average nonnegative.

The QDF $Q_\Psi(\ell)$ induced by $\Psi \in \mathbb{R}^{2m \times m}[\zeta, \eta]$ in (13) is called a *storage function for* $Q_\Phi(\ell)$ if the *dissipation inequality*

$$\nabla Q_\Psi(\ell)(t_1, t_2) \leq Q_\Phi(\ell)(t_1, t_2), \quad \forall \ell \in (\mathbb{R}^m)^{\mathbb{Z}^2}, (t_1, t_2) \in \mathbb{Z}^2$$

holds. Moreover, the QDF $Q_\Delta(\ell)$ induced by $\Delta \in \mathbb{R}_s^{m \times m}[\zeta, \eta]$ is called a *dissipation rate for* $Q_\Phi(\ell)$ if

$$\sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_\Phi(\ell)(t_1, t_2) = \sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_\Delta(\ell)(t_1, t_2)$$

and $Q_\Delta(\ell)(t_1, t_2) \geq 0$ hold for all $\ell \in l_2^m$ and $(t_1, t_2) \in \mathbb{Z}^2$. There is a one-to-one relation between a storage function $Q_\Psi(\ell)$ and a dissipation rate $Q_\Delta(\ell)$ defined by

$$\begin{aligned} \nabla Q_\Psi(\ell)(t_1, t_2) &= Q_\Phi(\ell)(t_1, t_2) - Q_\Delta(\ell)(t_1, t_2), \\ &\forall \ell \in (\mathbb{R}^m)^{\mathbb{Z}^2}, (t_1, t_2) \in \mathbb{Z}^2. \end{aligned} \quad (15)$$

The equality (15) is called the *dissipation equality*.

We give a necessary and sufficient condition for dissipativeness using QDFs as a main result.

Theorem 2: Let the controllable behavior \mathfrak{B} be given by (2). Suppose that \mathfrak{B} is represented by an image representation (3). Define $\Phi \in \mathbb{R}_s^{m \times m}[\zeta, \eta]$ by (12) for a given $\Pi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$. The following statements are equivalent.

- (i) The behavior \mathfrak{B} is dissipative with respect to the supply rate $Q_\Pi(w)$.
- (ii) The QDF $Q_\Phi(\ell)$ is average nonnegative.
- (iii) The frequency domain inequality

$$\partial\Phi(e^{i\omega_1}, e^{i\omega_2}) \geq 0 \quad (16)$$

holds for all $\omega_1, \omega_2 \in [0, 2\pi)$.

(iv) The QDF $Q_{\Phi}(\ell)$ admits a storage function and a dissipation rate.

Proof: (i) \Leftrightarrow (ii) The equivalence of is clear from the definition of $\Phi(\zeta, \eta)$ in (12).

(ii) \Leftrightarrow (iii) The equivalence follows from an analogous discussion to [3] by using the discrete-time Parseval's equality.

(iii) \Leftrightarrow (i) We can derive a four-variable polynomial matrix equation from the dissipation equality in terms of QDFs. By letting $\zeta_1 = e^{-i\omega_1}$, $\zeta_2 = e^{-i\omega_2}$, $\eta_1 = e^{i\omega_1}$ and $\eta_2 = e^{i\omega_2}$ in this equation, we obtain (16).

(vi) \Rightarrow (i) The proof is straightforward.

(i) \Rightarrow (iv) We see that $\Phi(\zeta, \eta)$ can be decomposed to $\Phi(\zeta, \eta) = \Gamma_2(\zeta_2)^\top \bar{\Phi}_1(\zeta_1, \eta_1) \Gamma_2(\eta_2)$ for some $\bar{\Phi}_1 \in \mathbb{R}_s^{(N_2+1)m \times (N_2+1)m}[\zeta_1, \eta_1]$, where $\Gamma_k(\xi_k) := \text{col}(I_m, \xi_k I_m, \dots, \xi_k^{N_k} I_m)$ ($k = 1, 2$) and $\mathbb{R}_s^{q \times q}[\zeta_1, \eta_1]$ denotes the set of $q \times q$ symmetric two-variable polynomial matrices in the indeterminates ζ_1, η_1 . Define the latent variable $\bar{\ell}_1 \in (\mathbb{R}^{(N_2+1)m})^{\mathbb{Z}^2}$ by $\bar{\ell}_1 = \Gamma_2(\sigma_2)\ell$. Then, we have $Q_{\Phi}(\ell)(t_1, t_2) = Q_{\bar{\Phi}_1}(\bar{\ell}_1)(t_1, t_2)$, $\forall (t_1, t_2) \in \mathbb{Z}^2$. Substituting this equality to (14), we get

$$\sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_{\Phi}(\ell)(t_1, t_2) = \sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_{\bar{\Phi}_1}(\bar{\ell}_1)(t_1, t_2) \geq 0, \forall \bar{\ell}_1 \in l_2^{(N_2+1)m}. \quad (17)$$

We now prove that the inequality

$$\sum_{t_1=-\infty}^{+\infty} Q_{\bar{\Phi}_1}(\bar{\ell}_1)(t_1, t_2) \geq 0, \forall \bar{\ell}_1(\cdot, t_2) \in (\mathbb{R}^{(N_2+1)m})^{\mathbb{Z}} \quad (18)$$

holds for an arbitrarily fixed $t_2 \in \mathbb{Z}$. Note that we regard $\bar{\ell}_1(\cdot, t_2) \in (\mathbb{R}^{(N_2+1)m})^{\mathbb{Z}}$ as a trajectory of a 1-D behavior.

Let us consider the trajectory $\tilde{\ell} \in l_2^{(N_2+1)m}$ defined by $\tilde{\ell}(t_1, t_2) = \bar{\ell}(T, t_2)$ for $t_1 = T$ and $\tilde{\ell}(t_1, t_2) = 0$ for $t_1 \neq T$. Assume that this trajectory satisfies $\sum_{t_2=-\infty}^{+\infty} \|\tilde{\ell}(T, t_2)\|^2 < \infty$ and $Q_{\bar{\Phi}_1}(\tilde{\ell})(T, t_2) < 0$ at $t_1 = T$. Then, the summation of $Q_{\bar{\Phi}_1}(\bar{\ell}_1)(t_1, t_2)$ with respect to t_1 satisfies

$$\sum_{t_1=-\infty}^{+\infty} Q_{\bar{\Phi}_1}(\bar{\ell}_1)(t_1, t_2) = Q_{\bar{\Phi}_1}(\tilde{\ell})(T, t_2) < 0.$$

Summing up the above inequality from $t_2 = -\infty$ to $t_2 = +\infty$, we get $\sum_{t_1=-\infty}^{+\infty} \sum_{t_2=-\infty}^{+\infty} Q_{\bar{\Phi}_1}(\bar{\ell}_1)(t_1, t_2) < 0$. This is a contradiction, since $\bar{\ell}_1(t_1, t_2)$ in (17) can be arbitrarily chosen. Thus, we obtain (18).

By applying Proposition 3.3 in [1] to (18), there exist $\bar{\Psi}_1, \bar{\Delta}_1 \in \mathbb{R}_s^{(N_2+1)m \times (N_2+1)m}[\zeta_1, \eta_1]$ satisfying the two-variable polynomial matrix equation

$$(\zeta_1 \eta_1 - 1) \bar{\Psi}_1(\zeta_1, \eta_1) = \bar{\Phi}_1(\zeta_1, \eta_1) - \bar{\Delta}_1(\zeta_1, \eta_1) \quad (19)$$

and the inequality

$$Q_{\bar{\Delta}_1}(\bar{\ell}_1)(t_1, t_2) \geq 0, \forall \bar{\ell}_1(\cdot, t_2) \in (\mathbb{R}^{(N_2+1)m})^{\mathbb{Z}}. \quad (20)$$

Premultiplying (19) by $\Gamma_2(\zeta_2)^\top$ and postmultiplying (19) by $\Gamma_2(\eta_2)$, we get the four-variable polynomial matrix equation

$$(\zeta_1 \eta_1 - 1) \Psi'_1(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta'_1(\zeta, \eta), \quad (21)$$

where $\Psi'_1 \in \mathbb{R}_s^{m \times m}[\zeta, \eta]$ is defined by $\Psi'_1(\zeta, \eta) := \Gamma_2(\zeta_2)^\top \bar{\Psi}_1(\zeta_1, \eta_1) \Gamma_2(\eta_2)$ and $\Delta'_1 \in \mathbb{R}_s^{m \times m}[\zeta, \eta]$ is defined by the similar way. In addition, it follows from (20) that

$$Q_{\Delta'_1}(\ell)(t_1, t_2) \geq 0, \forall \ell \in (\mathbb{R}^m)^{\mathbb{Z}^2}, (t_1, t_2) \in \mathbb{Z}^2. \quad (22)$$

By a similar discussion, we can prove the existence of $\Psi'_2 \in \mathbb{R}_s^{m \times m}[\zeta, \eta]$ and $\Delta'_2 \in \mathbb{R}_s^{m \times m}[\zeta, \eta]$ satisfying

$$(\zeta_2 \eta_2 - 1) \Psi'_2(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta'_2(\zeta, \eta) \quad (23)$$

and

$$Q_{\Delta'_2}(\ell)(t_1, t_2) \geq 0, \forall \ell \in (\mathbb{R}^m)^{\mathbb{Z}^2}, (t_1, t_2) \in \mathbb{Z}^2. \quad (24)$$

Taking the sum of (21) and (23), we get

$$\nabla \Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta), \quad (25)$$

where $\Psi \in \mathbb{R}^{2m \times m}[\zeta, \eta]$ is defined by (13) with $\Psi_1(\zeta, \eta) := \frac{1}{2} \Psi'_1(\zeta, \eta)$, $\Psi_2(\zeta, \eta) := \frac{1}{2} \Psi'_2(\zeta, \eta)$ and $\Delta \in \mathbb{R}_s^{m \times m}[\zeta, \eta]$ is given by $\Delta(\zeta, \eta) := \frac{1}{2} (\Delta'_1(\zeta, \eta) + \Delta'_2(\zeta, \eta))$. From (18) and (22), (24), we also have $Q_{\Delta}(\ell)(t_1, t_2) \geq 0$ for all $\ell \in (\mathbb{R}^m)^{\mathbb{Z}^2}$ and $(t_1, t_2) \in \mathbb{Z}^2$. Thus, the statement (iv) follows. \blacksquare

Remark 2: We emphasize a theoretical contribution of this paper in this remark.

In the previous results for 1-D behaviors [7][1], a dissipation rate is constructed by a polynomial spectral factor of the polynomial defined by the supply rate. However, it was pointed out in [3] that we cannot necessarily choose the factor as polynomial matrices for the n -D case. This paper avoids this difficulty by separating the dissipativeness of 2-D behaviors into a pair of average nonnegativity of 1-D behaviors with respect to t_1 and t_2 . See the derivation of the inequality (18) in the proof of Theorem 2 (i) \Rightarrow (iv).

V. CONCLUSIONS

In this paper, we have derived necessary and sufficient conditions for losslessness and dissipativeness of 2-D discrete-time behaviors based on QDFs as a main result.

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