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On first order state constrained optimal control problems

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Abstract— **We show that exact penalization techniques can be applied to optimal control problems with state constraints under a hard to verify hypothesis. Investigating conditions implying our hypothetical hypothesis we discuss some recent theoretical results on regularity of multipliers for optimal control problem involving first order state constraints. We show by an example that known conditions asserting regularity of the multipliers do not prevent the appearance of atoms in the multiplier measure. Our accompanying example is treated both numerically and analytically. Extension to cover problems with additional mixed state constraints is also discussed.**

I. INTRODUCTION

Optimal control problems with state constraints have been a challenging subject since the very birth of optimal control in the end of the 1950's. There is a vast literature addressing many issues of interest concerning the presence of state constraints, among those questions relating to the normality and the degeneracy of the maximum principle as well as the regularity of the optimal control and multipliers (see, for example, [2], [5], [1], [14], [15], [16], [19], [17] and references within). The presence of measure measures as multipliers associated with the state constraint in the Maximum Principle (see for example [24]) is a source of hardship both analytically and numerical. It is thus natural to ask if there exists any class of problem with state constraints where such measures are *well behaved* in the sense that they can be absolutely continuous with respect to the Lebesgue measure. In this paper, we investigate such question. We first explore exact penalization techniques to see how the maximum principle would look like and we discuss the difficulties concerning the validation of such result. Notably, we show that a Maximum Principle without measure would be possible if a certain condition were valid. Failing to validate such handy result we turn the analysis around. Looking to results asserting some regularity of the multipliers, we look at one of the simplest problems of state constrained optimal control problem. For such problems we briefly review the literature where some regularity of the measures is proved and we apply them to a specific problem arising from the control of the spreading of infectious diseases. An important feature of this problem is that it has a *first order* state constraint. Taking into account numerical results, we show that the measure associated with the state constraint is absolutely continuous inside the interval but has an atom at the end point. We go a step further discussing problems with additional mixed constraints.

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Notation

For g in $\mathbb{R}^m$, inequalities like $g \leq 0$ are interpreted componentwise. Here and throughout, $\bar{B}$ represents the closed unit ball centered at the origin regardless of the dimension of the underlying space and $|\cdot|$ the Euclidean norm or the induced matrix norm on $\mathbb{R}^{p\times q}$.

A function $h\colon [a,b] \to \mathbb{R}^p$ lies in $W^{1,1}([a,b];\mathbb{R}^p)$ if and only if it is absolutely continuous; in $L^1([a,b];\mathbb{R}^p)$ iff it is integrable; and in $L^\infty([a,b];\mathbb{R}^p)$ iff it is essentially bounded. The norm of $L^\infty([a,b];\mathbb{R}^p)$ is $\|\cdot\|_\infty$.

We make use of standard concepts from nonsmooth analysis. Let $A \subset \mathbb{R}^k$ be a closed set with $\bar{x} \in A$. The *proximal normal cone to A at $\bar{x}$* is denoted by $N_A^P(\bar{x})$, while $N_A^L(\bar{x})$ denotes the *limiting normal cone* and $N_A^C(\bar{x})$ is the *Clarke normal cone*.

Given a lower semicontinuous function $f\colon \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ and a point $\bar{x} \in \mathbb{R}^k$ where $f(\bar{x}) < +\infty$, $\partial^L f(\bar{x})$ denotes the *limiting subdifferential* of f at $\bar{x}$. When the function f is Lipschitz continuous near x, the convex hull of the limiting subdifferential, co $\partial^L f(x)$, coincides with the *(Clarke) subdifferential* $\partial^C f(\bar{x})$.

II. EXACT PENALIZATION

Consider the following problem

$$(P)\quad \left\{\begin{array}{l} \text{Minimize } l(x(a),x(b)) \\ \text{subject to} \\ \quad \dot{x}(t) = f(t,x(t),u(t)) \quad \text{a.e. } t\in[a,b] \\ \quad h(x(t)) \leq 0 \quad \text{ for all } \ t\in[a,b] \\ \quad u(t) \in U(t) \quad \text{a.e. } t \in [a,b] \\ \quad (x(a),x(b)) \in E. \end{array}\right.$$

The function $f:\ \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^n$ describes the system dynamics and $h:\ \mathbb{R}^n \to \mathbb{R}$ is the functional defining the pure state constraint. Furthermore, the set $E \subset \mathbb{R}^n\times\mathbb{R}^n$ and $l:\ \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ specify the endpoint constraints and cost. The multifunction $t \longrightarrow U(t)$ defines the pointwise set control constraints. Observe that we introduce a simplification by assuming that h is a function of x alone.

This problem involves measurable control functions u and absolutely continuous state function x. A pair (x,u) is called an admissible process if it satisfies the constraints of the problem with finite cost.

We say that the process $(\bar{x},\bar{u})$ is a *strong local minimum* if, for some $\varepsilon > 0$, it minimizes the cost over admissible

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processes (x,u) such that $|x(t)-\bar{x}(t)| \le \varepsilon$ for all $t \in [a,b]$.

We consider the following *basic hypotheses* on the problem data throughout which make reference to some process $(\bar{x},\bar{u})$ and a scalar $\varepsilon > 0$:

H1 The function $t \longrightarrow f(t,x,u)$ is $\mathcal{L}$ measurable for all x and u.

H2 There exists a constant $K_l > 0$ such that

$$|l(x_a,x_b) - l(x'_a,x'_b)| \le K_l|(x_a,x_b)-(x'_a,x'_b)|$$

for all (x_a,x_b), (x'_a,x'_b) such that $x_a, x'_a \in \bar{x}(a)+\varepsilon\bar{B}$, $x_b, x'_b \in \bar{x}(b)+\varepsilon\bar{B}$.

H3 The set E is closed.

H4 The function h is continuously differentiable on and $\nabla h(x) \neq 0$ for any x such that $h(x)=0$.

H5 There exist constants k_x^f and k_x^fsuch that all $u,u' \in \mathbb{R}^k$ and all $x, x' \in \bar{x}(t)+\varepsilon B$ we have

$$|f(t,x,u)-f(t,x',u')| \le k_x^f|x-x'| + k_u^f|u-u'|$$

for almost every $t \in [a,b]$.

H6 The graph of the multifunction U is a Borel set.

Set $\Phi := \{y \in \mathbb{R} :\ y \le 0\}$ and

$$S := \{x \in \mathbb{R}^n :\ h(x) \in \Phi\}. \tag{1}$$

Observe that $h(x) \le 0 \iff x \in S$. Consider the distance function

$$d_S(x) = \inf\{|x-x'| :\ x' \in S\}.$$

By definition of S we have

$$d_S(\bar{x}(t)) = 0 \Longleftrightarrow h(\bar{x}(t)) \le 0.$$

The set Φ is convex and thus $N_\Phi^L(y) = N_\Phi^C(y)$ for all $y \in \mathbb{R}$. If, for some $x \in \mathbb{R}^n$, we have $h(x) < 0$, then $N_\Phi^C(h(x)) = \{0\}$. If, however, $h(x)=0$, then

$$\xi \in N_\Phi^C(h(x)) \Longrightarrow \xi \ge 0.$$

Recall that (see [7], [24] for example)

$$\partial^C d_S(x) \subset N_S^C(x).$$

Then

$$\zeta \in \partial^C d_S(x) \Longrightarrow \zeta \in N_S^C(x).$$

By (H4), if $\alpha \in \mathbb{R}$ such that $\alpha \ge 0$ and $\partial^L \alpha h(x) = 0$, then $\alpha = 0$. It then follows from Proposition 4.1 in [10] that

$$\forall\ \zeta \in \partial^C d_S(x)\ \ \exists \alpha \in N_\Phi^C(h(x)) :\ \zeta = \alpha \nabla h(\bar{x}(t)). \tag{2}$$

This last finding will be important in the subsequence development.

We now add the following *hypothetical* assumption:

HH If $(\bar{x},\bar{u})$ is a strong minimum for (P), then there exists a constant $K > K_l$ such that $(\bar{x},\bar{u})$ is a strong minimum for the following problem

$$(Q) \quad \begin{cases} \text{Minimize } l(x(a),x(b)) + K\displaystyle\int_a^b d_S(x(t))dt \\ \text{subject to} \\ \quad \dot{x}(t) = f(t,x(t),u(t)) \quad \text{a.e. } t \in [a,b] \\ \quad u(t) \in U(t) \quad \text{a.e. } t \in [a,b] \\ \quad (x(a),x(b)) \in E. \end{cases}$$

where the set S is as defined in (1).

Observe that (Q) is an optimal problem without state constraints: the state constraint $h(x(t)) \le 0$ in (P) is incorporated in the cost function of (Q) via the integral of the distance function d_S.

We call (HH) a *hypothetical* assumption since what we would like to have is an assumption implying (HH). Let us see what we would need to guarantee (HH). It is a simple matter to see that $(\bar{x},\bar{u})$ is an admissible solution to (Q). Moreover, any admissible process (z,v) for (P) is an admissible process for (Q).

Seeking a contradiction we suppose that $(\bar{x},\bar{u})$ is not a strong minimum to (Q). First, observe that, since $h(\bar{x}(t)) \le 0$ for all $t \in [a,b]$, we have $K\displaystyle\int_a^b d_S(\bar{x}(t))dt = 0$. Take now (x',u') to be an admissible process for (Q) such that

$$l(x'(a),x'(b)) + K\int_a^b d_S(x'(t))dt < l(\bar{x}(a),\bar{x}(b))$$

and set

$$\rho = l(\bar{x}(a),\bar{x}(b)) - l(x'(a),x'(b)) - K\int_a^b d_S(x'(t))dt.$$

It is then obvious that $\rho > 0$. Choose some $\delta \in (0, \frac{\rho}{2K})$. Then we have

$$0 < K\delta \le \frac{\rho}{2} < \rho.$$

Consequently

$$0 < l(\bar{x}(a),\bar{x}(b)) - K\delta - l(x'(a),x'(b)) - K\int_a^b d_S(x'(t))dt.$$

It follows that

$$l(x'(a),x'(b)) + K\int_a^b d_S(x'(t))dt < l(\bar{x}(a),\bar{x}(b)) - K\delta.$$

Suppose now that that there exists an admissible process (z,v) for (P) such that

$$\max_{t\in[a,b]}\{|z(t)-x'(t)|\} \le \frac{K}{2}\int_a^b d_S(x'(t))dt. \tag{3}$$

We know that

$$\Big|(z(a),z(b)) - (x'(a),x'(b))\Big| \le 2\max_{t\in[a,b]}\{|z(t)-x'(t)|\}$$

and since l is Lipschitz we have

$$l(z(a),z(b)) - l(x'(a),x'(b)) \le K_l\Big|(z(a),z(b)) - (x'(a),x'(b))\Big|.$$

Thus, assuming $K > K_l$ we have

$$
\begin{gathered}
l(z(a), z(b)) - l(x'(a), x'(b)) \leq \\
K\Big|(z(a), z(b)) - (x'(a), x'(b))\Big| \leq \\
K \int_a^b d_S(x'(t))dt < K \int_a^b d_S(x'(t))dt + K\delta < \\
K \int_a^b d_S(x'(t))dt + \rho = \\
K \int_a^b d_S(x'(t))dt + l(\bar{x}(a), \bar{x}(b)) - l(x'(a), x'(b)) \\
-K \int_a^b d_S(x'(t))dt = \\
l(\bar{x}(a), \bar{x}(b)) - l(x'(a), x'(b))
\end{gathered}
$$

and we deduce that

$$l(z(a), z(b)) \leq l(\bar{x}(a), \bar{x}(b))$$

contradicting the optimality of $(\bar{x}, \bar{u})$.

We deduce from the above that if for any admissible process (x', u') for (Q) there exists an admissible process (z, v) for (P) satisfying ((3), then (HH) would hold. However, such condition involving (3) is not satisfied in general.

Observe that (3) can be written as

$$\|z - x'\|_\infty \leq \frac{K}{2} \int_a^b d_S(x'(t))dt. \tag{4}$$

The existence of an admissible process (z, v) for (P) satisfying conditions somewhat similar to (4) has been vastly explored in the literature (see, for example, [17], [2], [3], [4]). However, no conditions involving $\displaystyle\int_a^b d_S(x'(t))dt$ are known to hold. Nevertheless and as we can see next, if some conditions on the data of (P) would imply (4), they would be of use as we illustrate next.

A. Hypothetical Theorem

Now let us work under the assumptions (H1)–(H6) and (HH). The following result can be obtained by simply applying to (Q) the nonsmooth Maximum Principle (see, for example, [24]) and appealing to (2):

Let $(\bar{x}, \bar{u})$ be a strong minimum for (P). Then then there exist an absolutely continuous function p, a measurable function ξ and a scalar $\lambda \geq 0$ such that

(i) $$||p||_\infty + \lambda > 0,$$

(ii) $$-\dot{p}(t) \in \partial_x^C \langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - \lambda\xi(t)\nabla h(\bar{x}(t)) \quad \text{a.e.},$$

(iii) $$u \in U(t) \Longrightarrow \langle p(t), f(t, \bar{x}(t), u)\rangle \leq \langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle \quad \text{a.e.},$$

(iv) $$(p(a), -p(b)) \in N_E^L(\bar{x}(a), \bar{x}(b)) + \lambda\partial^L l(\bar{x}(a), \bar{x}(b)),$$

(v) $$\xi(t) \geq 0 \text{ and } \xi(t)h(\bar{x}(t)) = 0 \quad \text{a.e.}$$

Notably no measure is present in the above conditions.

III. A FIRST ORDER PROBLEM

Let us now turn to autonomous problems of the form

$$
(\text{FO}) \quad \begin{cases}
\text{Minimize } \displaystyle\int_a^b \langle c, x\rangle + u^2\, dt \\
\text{subject to} \\
\quad \dot{x}(t) = f(x(t)) + g(x(t))u(t) \text{ a.e.}t, \\
\quad h(x(t)) \leq 0 \quad \forall t, \\
\quad u(t) \in U \quad \text{a.e.}t, \\
\quad x(a) = x_a,
\end{cases}
$$

where $c \in \mathbb{R}^n$, u is a scalar, U is a compact set in $\mathbb{R}$ and $\nabla h(x) \neq 0$ whenever $h(x) = 0$. Here, as before, we assume that h is a scalar valued function.

Before proceeding, let us briefly review some concepts on the state constraint appearing in problem (FO) that will be important in our setting. A boundary interval for the state constraint along a trajectory x of (FO) is an interval $[t_0^b, t_1^b] \subset [a, b]$ if it is the maximal interval where $h(x(t)) = 0 \quad \forall t \in [t_0^b, t_1^b]$. The point t_0^b and t_1^b are called *entry point* and *exit point*. Any interval $I \subset [a, b]$ is an *interior interval* if $h(x(t) < 0 \quad \forall t \in I$. A point $\sigma \in [a, b]$ is a *contact point* for x if it is an isolated point such that $h(x(\sigma)) = 0$.

Problem (FO) has one state constraint. Let $(\bar{x}, \bar{u})$ be local strong minimum for (FO) and assume that our conditions H1-H6 are satisfied. Theorem 9.3.1 in [24] asserts that there exist an absolutely continuous function p, a scalar λ, a measure $\mu \in C^\oplus([a, b])$ such that

(i) $(p, \lambda, \mu) \neq (0, 0, 0)$;

(ii) $-\dot{p}(t) = f_x^T(\bar{x}(t))q(t) + \bar{u}(t)g_x^T(\bar{x}(t))q(t) - \lambda c$;

(iii) for all $u \in U$, $\langle g(\bar{x}(t))\bar{u}(t), q(t)\rangle - \lambda\bar{u}^2(t) \geq \langle g(\bar{x}(t))u, q(t)\rangle - \lambda u^2$,

(iv) $-q(b) = 0$;

(v) $supp\{\mu\} \subset \{t : \ h(x^*(t)) = 0\}$.

where

$$
\begin{aligned}
q(t) &= p(t) + \textstyle\int_{[a,t)} \nabla h(\bar{x}(s))\, \mu(ds), \\
q(b) &= p(b) + \textstyle\int_{[a,b]} \nabla h(\bar{x}(s))\, \mu(ds).
\end{aligned}
$$

If the above conditions hold with $\lambda = 1$, then we say that the problem is normal.

In [22] (see also [17] and the references within), conditions are derived for the problems of the type of (FO) to guarantee that the measure associated with the state constraint in the *normal* form of the maximum principle is regular (in the sense that it is absolutely continuous with respect to the Lebesgue measure in the interior of the interval $[a, b]$). One might think that conditions imposed to assert regularity of the adjoint variable, if satisfied, would also imply that our hypothetical assumption (HH) holds. However, we need to keep in mind that neither [22] nor [17] provide us with information about the possible behaviour at the points $t = a$ or $t = b$. If we knew a priori that $h(\bar{x}(a)) < 0$ and $h(\bar{x}(b)) < 0$, then, the regularity of the measure would follow. But no such guarantee exists as we illustrate next with a simple problem with one state constraint of first order recovered from [6].

A. An example with state constraints

We want to determine a vaccination policy to control the spreading of a generic infectious diseases in a certain population. Let us consider a "SEIR" model. This is a compartmental model dividing the total population N into four different compartments relevant to the epidemic. Those compartments are susceptible (S), exposed (E), infectious (I), and recovered (immune by vaccination) (R). We look at the evolution of the disease over a certain period of time T with parameters describing the population and the disease transmission constant over a period of time T. Taking into account certain assumptions on the population and the disease transmission (see [20] for a more complete description), considering a simple cost and requiring that the susceptible population is to be bounded at each instant of time, we are led to following the problem (P_S):

$$\begin{cases} \text{Minimize } \displaystyle\int_0^T \left(AI(t) + u^2(t)\right) dt \\ \text{subject to} \\ \begin{aligned} \dot S(t) &= bN(t) - dS(t) - cS(t)I(t) - u(t)S(t), \\ \dot E(t) &= cS(t)I(t) - (e+d)E(t), \\ \dot I(t) &= eE(t) - (g+a+d)I(t), \\ \dot N(t) &= (b-d)N(t) - aI(t), \\ S(t) &\leq S_{max}, \\ u(t) &\in [0,1] \text{ a.e.} t, \\ S(0) &= S_0,\ \ E(0) = E_0, \\ I(0) &= I_0,\ \ N(0) = N_0. \end{aligned} \end{cases}$$

The above problem has the form of problem (FO) as shown in [6]. To see this set

$$x(t) = (S(t), E(t), I(t), N(t)),$$
$$\tilde A = (0, 0, A, 0), \quad C = (1, 0, 0, 0).$$

and for a convenient choice of matrices A_1 and B, define where $f_1(x) = A_1 x + c(-SI, SI, 0, 0)^T$, (here c is some parameter) $g(x) = Bx$ and $h(x) = \langle C, x\rangle - S_{max} = S - S_{max}$ for some fixed $S_{max} > S(0)$.

Now we concentrate on the necessary conditions given by (i))-(v) above. Consider $q = (q_s, q_e, q_i, q_n)$ and analogously $p = (p_s, p_e, p_i, p_n)$. It is an easy task to show that in the region of interest Theorem 4.1 in [21] holds. Thus conditions (i)-(v) with $\lambda = 1$. Applying such conditions to (P_S) we deduce that

$$\bar u(t) = \max\left\{0, \min\left\{1, -\frac{q_s(t)S^*(t)}{2}\right\}\right\}. \qquad (5)$$

Since

$$q(t) = p(t) + \int_{[0,t)} \nabla h(x^*(t))\,\mu(ds),$$

and

$$\nabla h(x^*(t)) = (1, 0, 0, 0),$$

we obtain

$$\int_{[0,t)} (1, 0, 0, 0)\,\mu(ds) = \left(\int_{[0,t)} \mu(ds), 0, 0, 0\right)$$

Then we get

$$q_s(t) = p_s(t) + \int_{[0,t)} \mu(ds),$$

$q_e(t) = p_e(t)$, $q_i(t) = p_i(t)$, $q_n(t) = p_n(t)$.

Appealing to the main results of [22] we conclude that μ is absolutely continuous w.r.t. Lebesgue measure. Hence, there exists an integrable function ν such that

$$\int_0^t \nu(s)\,ds = \int_{[0,t)} \mu(ds)$$

and q is absolutely continuous on $[0, T[$ with

$$\dot q_s(t) = \dot p_s(t) + \nu(t).$$

It is now a simple matter to see that for $t \in [0, T[$,

$$\dot q_s(t) = (d + cI^*(t) + u^*(t))q_s(t) - \nu(t) + cI^*(t)q_e(t).$$

We do not know whether q_s has a jump at $t = T$ or not. However, the maximum principle tells us that $q_s(T) = 0$ and that we have $p_e(t) = q_e(t)$, $p_i(t) = q_i(t)$, $p_n(t) = q_n(t)$ and $p_e(T) = p_i(T) = p_n(T) = 0$.

To see if q_s has a jump on T or not, we treat our problem numerically. To do our simulations we use the Imperial College London Optimal Control Software – ICLOCS – version $0.1b$. ICLOCS calls IPOPT – Interior Point OPTimizer – an open-source software package for large-scale nonlinear optimization. Treating our numerical data we show in [18] that the numerical solution does satisfy the maximum principle and the state constraint is active at the end point $t = T$ (see Figure 1).

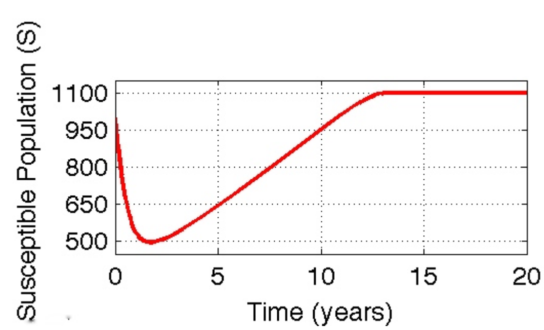

Fig. 1. The state (S) for (P_S)

Moreover, checking the the numerical multipliers, we deduce that the measures (see also [18]) does indeed have an atom. This is of interest because of the simple structure of the state constraint of (P_S). Indeed, this state constraint is of *first order*.

B. Example with State and Mixed constraints

As in [6], we can now add a mixed constraint of the form $S(t)u(t) \leq V_0$ to (P_S) creating a new problem, (P_{MS}), with both mixed and state constraints.

Appealing to appropriate necessary conditions for (P_{MS}) and adapting the proof of known results such as Theorem 4.1 in [21] we can show that, as before, (P_{MS}) is normal. Regularity of the adjoint variable can also be guaranteed, adapting, for example the proofs, of [22]. Hence, once more, we have the regularity of the measure in $[0, T[$. However, our numerical findings also show that the state constraint is active at the end point and that the measure has an atom at $t = T$ as shown in Figures 2 and 3.

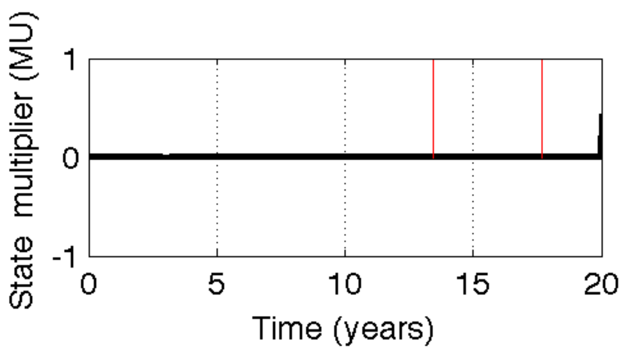

Fig. 2. The state multiplier exhibiting a jump at the end point for (P_{MS})

Once more, the pathological aspect is the fact that the state constraint is active at the end point.

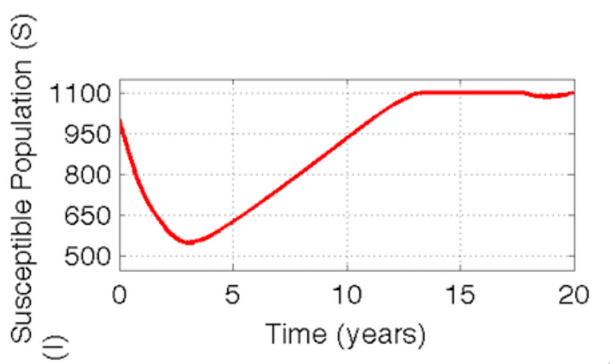

Fig. 3. The optimal state (S) for (P_{MS}).

IV. Conclusions

It is important to identify a class of state constrained problems with regular adjoint variables, since the conclusions of the maximum principle under such circumstances would be easy to apply. We investigate here some possible conditions under which this would hold. Moreover, we point out that when the state constraint is not active at the end points, such class of problems coincide with the class of problems to which regularity conditions derived in [17] and [22] hold. However, even for problems with a simple structure, it is not an easy task to guarantee that the state constraint is inactive on end points. We present an example with first order state constraints where the state constraint is active on the end point and the adjoint variable does have a jump there. We believe that the analysis of known results on regularity is not affected under addition of a mixed constraint (this will be the focus of future work). Remarkably, adding mixed constraint to our example problem does not change the behaviour of our adjoint variable.

Our examples have active constraints on the end point but do have regularity of the measure at any point $t \neq T$. Taking into account the literature on nondegeneracy of the maximum principle, we hope that nondegeneracy conditions at the end point together with the known regularity conditions for problems in the form of (FO) may be of help in the quest for conditions implying our hypothetical assumption (HH).

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