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Different aspects of the controllability problem for rotating Timoshenko beams*

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***Abstract*— We consider the null-controllability problem for rotating Timoshenko beams. We present a short survey of the results on different forms of reachability conditions and determining the minimal exact controllability time. The main result concerns the problem of optimal control in the sense of minimal energy. We give a new numerical approach of finding the optimal control. The method is based on the analysis of the specific properties of the moment problem corresponding to the problem of rest-to-rest controllability.**

I. ROTATING TIMOSHENKO BEAMS MODELS. SPECTRAL PROPERTIES

The problem of control of vibrating beams has been the subject of several investigations during last decades. A number of publications concentrate on the Euler beam model [4], [6], [10], [12], [13]. At the same time there appear more and more works concerning Timoshenko beam model [15], [16], [17], [18], [19], [21], [29], [31]. In [32] the nonlinear model for a rotating Timoshenko beam in a horizontal plane was derived and the linearization of the problem in the case of a slowly rotating beam was given. This particular model was picked up by Krabs and Sklyar and studied in [16], [17], [18]. In the monograph [18] the detailed spectral analysis of the operators associated with Timoshenko beam model was given (for the case of non-homogenous beam see also [22]). This allowed statement and studying – for this model – major problems concerning control theory: exact controllability, approximate controllability and stabilizability.

We recall the dimension-free model from [16], [22] of a Timoshenko beam rotating in a horizontal plane, whose left end is clamped into the disk of a driving motor. Let $r > 0$ be the radius of the disk and let $\theta = \theta(t)$ be the rotation angle as a function of time $t \geq 0$. By $w(x,t)$ we denote the deflection of the center line of the beam at the location $x \in [0,1]$ (the length of the beam is assumed to be 1) and the time $t \geq 0$, by $\xi(x,t)$ the rotation angle of the cross section area at x and t. We also set $E(x)$ for the flexural rigidity of the cross-section, $K(x)$ – shear stiffness, $\varrho(x)$ – mass of the cross section and $R(x)$ – rotary inertia; all of the above functions are assumed to be real, bounded from above and bounded from below by a positive number, their first and second derivatives are also bounded. If we assume the rotation to be slow, then w and ξ are governed by the two following differential equations

$$\begin{aligned} &\varrho(x)\ddot{w}(x,t) - \big(K(x)(w'(x,t) + \xi(x,t))\big)' \\ &\quad = -\ddot{\theta}(t)\varrho(x)(x+r), \\ &R(x)\ddot{\xi}(x,t) - (E(x)\xi'(x,t))' + K(x)(w'(x,t) + \xi(x,t)) \\ &\quad = \ddot{\theta}(t)R(x) \end{aligned} \tag{1}$$

with $x \in (0,1)$ and $t > 0$, where $\dot{y}(x,t) = \frac{\partial}{\partial t}y(x,t)$, $y'(x,t) = \frac{\partial}{\partial x}y(x,t)$. In addition we have boundary conditions given by

$$\begin{aligned} &w(0,t) = \xi(0,t) = 0, \\ &w'(1,t) + \xi(1,t) = \xi'(1,t) = 0 \end{aligned} \tag{2}$$

for $t \geq 0$, which mean that there is no deformation at the clamped end and the energy balance law holds at the other end.

Let $A : D(A) \to H = L^2\big((0,1),\mathbb{R}^2\big)$ be an operator given by $A\binom{y}{z} = \binom{-\frac{1}{\varrho}(K(y'-z))'}{-\frac{1}{R}((Ez')' - K(y'+z))}$ for $\binom{y}{z} \in D(A)$, where

$$D(A) = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in H^2((0,1),\mathbb{R}^2) \;\middle|\; \begin{array}{l} y(0) = z(0) = 0, \\ y'(1) + z(1) = 0, \\ z'(1) = 0 \end{array} \right\},$$

where the space H is endowed with inner product

$$\begin{aligned} \left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle &= \int_0^1 \varrho(x) y_1(x)\overline{y_2(x)}dx \\ &+ \int_0^1 R(x) z_1(x)\overline{z_2(x)}dx. \end{aligned}$$

Defining $b(x) = \binom{-r-x}{1}$ and $\binom{f_1(x,t)}{f_2(x,t)} = b(x)\ddot{\theta}(t)$ we rewrite (1) – (2) in the form of the operator model equation

$$\begin{pmatrix} \ddot{w}(\cdot,t) \\ \ddot{\xi}(\cdot,t) \end{pmatrix} + A\begin{pmatrix} w(\cdot,t) \\ \xi(\cdot,t) \end{pmatrix} = \begin{pmatrix} f_1(\cdot,t) \\ f_2(\cdot,t) \end{pmatrix} \tag{3}$$

for $t > 0$. In [16], [22], [23] it was found that A is positive, self-adjoint and has a sequence of simple (i.e. of multiplicity one) eigenvalues $\lambda_j \in \mathbb{R}$ such that

$$0 < \lambda_j \to \infty \quad \text{as } j \to \infty$$

and a corresponding orthonormal sequence of eigenelements $\binom{y_j}{z_j} \in D(A)$, $j \in \mathbb{N}$. Besides, it was proven that

$$\sqrt{\lambda_n}\cdot\left(\frac{2k-1}{2}\pi + \varepsilon_n\right)^{-1} = \begin{cases} \int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}dx \text{ for } n = 2k-1, \\ \int_0^1 \sqrt{\frac{R(x)}{E(x)}}dx \text{ for } n = 2k, \end{cases} \tag{4}$$

$\varepsilon_n = \mathcal{O}(\frac{1}{n})$.

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II. Minimal controllability time

Following [15] we consider the following

Problem of exact null-controllability: Assume that the beam is in the position of rest at time $t = 0$, i.e. the initial conditions

$$\begin{aligned} &w(x,0) = \dot{w}(x,0) = 0,\\ &\xi(x,0) = \dot{\xi}(x,0) = 0,\\ &\theta(0) = \dot{\theta}(0) = 0 \end{aligned} \tag{5}$$

with $x \in [0,1]$ are satisfied. Describe the set of the states $(w_T, \xi_T, \dot{w}_T, \dot{\xi}_T, \theta_T, \dot{\theta}_T)$ such that for given $T > 0$ we can find control $u = \ddot{\theta} \in L^2(0,T)$ satisfying

$$\begin{aligned} &\theta(0) = \dot{\theta}(0) = 0,\\ &\theta(T) = \theta_T,\\ &\dot{\theta}(T) = \dot{\theta}_T, \end{aligned}$$

and the solution (w,ξ) of (1) – (5) satisfies the end conditions

$$\begin{aligned} w(x,T) = w_T(x), \quad \xi(x,T) = \xi_T(x),\\ \dot{w}(x,T) = \dot{w}_T(x), \quad \dot{\xi}(x,T) = \dot{\xi}_T(x). \end{aligned}$$

The unique weak solution of equation (3) corresponding to initial conditions (5) is then given by

$$\begin{aligned} &\binom{w(x,t)}{\xi(x,t)}\\ &= \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \int_0^t \sin\sqrt{\lambda_j}(t-s) \left\langle \binom{f_1(\cdot,t)}{f_2(\cdot,t)}, \binom{y_j}{z_j} \right\rangle_H ds \binom{y_j}{z_j} \end{aligned}$$

for $x \in [0,1]$ and $t \geq 0$ and its time derivative reads

$$\begin{aligned} &\binom{\dot{w}(x,t)}{\dot{\xi}(x,t)}\\ &= \sum_{j=1}^{\infty} \int_0^t \cos\sqrt{\lambda_j}(t-s) \left\langle \binom{f_1(\cdot,t)}{f_2(\cdot,t)}, \binom{y_j}{z_j} \right\rangle_H ds \binom{y_j}{z_j} \end{aligned}$$

Hence the end conditions are equivalent to the following moment problem

$$\begin{aligned} \int_0^T \sin\sqrt{\lambda_j}(T-s)\ddot{\theta}(s)ds = \frac{\sqrt{\lambda_j}}{a_j} \left\langle \begin{pmatrix} w_T\\ \xi_T \end{pmatrix}, \begin{pmatrix} y_j\\ z_j \end{pmatrix} \right\rangle,\\ \int_0^T \cos\sqrt{\lambda_j}(T-s)\ddot{\theta}(s)ds = \frac{1}{a_j} \left\langle \begin{pmatrix} \dot{w}_T\\ \dot{\xi}_T \end{pmatrix}, \begin{pmatrix} y_j\\ z_j \end{pmatrix} \right\rangle, \end{aligned} \tag{6}$$

$j \in \mathbb{N}$, and

$$\begin{aligned} \int_0^T t\ddot{\theta}(T-t)dt = \theta_T,\\ \int_0^T \ddot{\theta}(T-t)dt = \dot{\theta}_T, \end{aligned} \tag{7}$$

where $a_j = \left\langle b, \binom{y_j}{z_j} \right\rangle$. Let $d_j = \frac{\left\langle \binom{\dot{w}_T}{\dot{\xi}_T}, \binom{y_j}{z_j} \right\rangle + i\sqrt{\lambda_j}\left\langle \binom{w_T}{\xi_T}, \binom{y_j}{z_j} \right\rangle}{a_j}$, and let us rewrite (6) in equivalent form of a trigonometric moment problem of Ullrich type, namely

$$\begin{aligned} \int_0^T e^{\sqrt{\lambda_j}(T-s)}\ddot{\theta}(s)ds = d_j,\\ \int_0^T e^{-\sqrt{\lambda_j}(T-s)}\ddot{\theta}(s)ds = \overline{d_j}, \end{aligned} \tag{8}$$

$j \in \mathbb{N}$. Now using Riesz basis results from [1] authors of [22] obtained the following result.

Theorem 1 *(see [22]). The problem of controllability from rest to rest is solvable if*

$$T > T_0 = 2\left(\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}dx + \int_0^1 \sqrt{\frac{R(x)}{E(x)}}dx \right).$$

The problem of null-controllability of the beam was also studied in [23] for specific conditions on values of integrals $\int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}dx$ and $\int_0^1 \sqrt{\frac{R(x)}{E(x)}}dx$.

III. Series convergence reachability conditions

Now assume that the beam is homogeneous, and consider the simplest case, namely $\varrho(x) = K(x) = R(x) = E(x) \equiv 1$. It is known that (8) has at most one solution if $T \leq T_0 = 4$ whereas if $T > T_0 = 4$ the solution of (8) is not unique (if it exists). It is well known that the problem of controllability of distributed parameter systems is often reduced to the corresponding trigonometric moment problem (see [20] as one of the first works concerning this issue). In [15] Korobov et al. showed that in the trigonometric moment problem corresponding to the model from [18] there appear two different but asymptotically close families of complex exponentials. Such a problem was earlier studied by Ullrich [30] in 80's in its abstract form. The conditions of solvability obtained by Ullrich are formulated as convergence of series of divided differences associated with the moment sequences. Later these results were essentially generalized in [2], [3] and others. Applying the Ullrich's approach the conditions of the exact controllability was obtained in the form of convergence of series of divided differences of sequences constructed by some parameters of model operators [15]. The result was developed for some more general model in [19].

Basing on results of Ullrich the authors of [15] obtained conditions of reachability of end states in terms of convergence of series:

Theorem 2 *(see [15]). Let $T > 4$. The state $(w_T, \xi_T, \dot{w}_T, \dot{\xi}_T, \theta_T, \dot{\theta}_T)$ is null-reachable by the virtue of the system* (3) *if and only if the condition*

$$\begin{aligned} &\sum_{j=1}^{\infty} \Big(|\mathrm{Im}d_j|^2 + |\mathrm{Re}d_j|^2\\ &+ \left| \frac{\mathrm{Im}d_{2j} - \mathrm{Im}d_{2j-1}}{\sqrt{\lambda_{2j}} - \sqrt{\lambda_{2j-1}}} \right|^2 + \left| \frac{\mathrm{Re}d_{2j} - \mathrm{Re}d_{2j-1}}{\sqrt{\lambda_{2j}} - \sqrt{\lambda_{2j-1}}} \right|^2 \Big) < \infty \end{aligned} \tag{9}$$

holds, where

$$\mathrm{Im}d_j = \sqrt{\lambda_j}\frac{\left\langle \binom{w_T}{\xi_T}, \binom{y_j}{z_j} \right\rangle_H}{\left\langle b, \binom{y_j}{z_j} \right\rangle_H}, \quad \mathrm{Re}d_j = \frac{\left\langle \binom{\dot{w}_T}{\dot{\xi}_T}, \binom{y_j}{z_j} \right\rangle_H}{\left\langle b, \binom{y_j}{z_j} \right\rangle_H}, \tag{10}$$

λ_j are eigenvalues of A and $\binom{y_j}{z_j} \in D(A)$ are corresponding eigenvectors, $j \in \mathbb{N}$.

Let $T = 4$. The state $(w_T, \xi_T, \dot{w}_T, \dot{\xi}_T, \theta_T, \dot{\theta}_T)$ is null-reachable by the virtue of the system (3) *if and only if the* *condition* (9) *holds and the end conditions*

$$\int_0^T t\ddot{\theta}(T-t)dt = \theta_T,$$
$$\int_0^T \ddot{\theta}(T-t)dt = \dot{\theta}_T$$

are satisfied, where $\ddot{\theta}(\cdot)$ *is the unique solution of* (8).

A similar results concerning homogeneous beam with physical parameters also constant, but not all equal 1, was obtained in [19].

IV. SMOOTHNESS OF END STATES CONDITIONS

We prove the following theorem which answers the question of the sense of Ullrich conditions in our problem:

Theorem 3. *(see [27]). Let* $T > 4$*. The state* $(w_T, \xi_T, \dot{w}_T, \dot{\xi}_T, \theta_T, \dot{\theta}_T)$ *is null-reachable by the virtue of the system* (3) *if and only if it fulfills the smoothness conditions of the form*

$$\begin{array}{l} w_T, \xi_T \in H^2(0,1), \\ \dot{w}_T, \dot{\xi}_T \in H^1(0,1), \\ \varphi_T \in H^3(0,1), \\ \dot{\varphi}_T \in H^2(0,1), \end{array}$$

and boundary conditions of the form

$$\begin{array}{l} w_T(0) = \xi_T(0) = 0, \\ w_T'(1) + \xi_T(1) = \xi_T'(1) = 0, \\ \varphi_T''(0) = 0, \end{array}$$

$$\begin{array}{l} \dot{w}_T(0) = \dot{\xi}_T(0) = 0, \\ \dot{\varphi}_T'(1) = \frac{1}{2}\dot{w}_T(1)\left(-r\cos\frac{1}{2} - \sin\frac{1}{2}\right) \\ \qquad -\frac{1}{2}\dot{\xi}_T(1)\left(-r\sin\frac{1}{2} + \cos\frac{1}{2}\right), \end{array}$$

where

$$\begin{array}{rcl} \varphi_T(x) &=& w_T(x)\left(-r\sin\frac{x}{2} + \cos\frac{x}{2}\right) \\ &+& \xi_T(x)\left(r\cos\frac{x}{2} + \sin\frac{x}{2}\right), \\ \dot{\varphi}_T(x) &=& \dot{w}_T(x)\left(-r\sin\frac{x}{2} + \cos\frac{x}{2}\right) \\ &+& \dot{\xi}_T(x)\left(r\cos\frac{x}{2} + \sin\frac{x}{2}\right). \end{array}$$

Let $T = 4$*. The state* $(w_T, \xi_T, \dot{w}_T, \dot{\xi}_T, \theta_T, \dot{\theta}_T)$ *is null-reachable by the virtue of the system* (3) *if and only if the smoothness and boundary conditions are fulfilled and the end conditions*

$$\int_0^T t\ddot{\theta}(T-t)dt = \theta_T,$$
$$\int_0^T \ddot{\theta}(T-t)dt = \dot{\theta}_T$$

are satisfied.

V. OPTIMAL CONTROL – NUMERICAL APPROACH

Let us assume again that all physical variable are constants, equal 1. Remind that for the minimal time $T = 4$ there exists only at most one control function steering from the preassigned position of rest to another, and for any $T > 4$ there exists an infinite number of such controls. Therefore for $T > 4$ it is natural to formulate problems of searching for special controls and the time interval cannot be shortened. In 2000 in [14] the numerical construction of a control function steering the beam from one position of rest to another was given for a time of movement large enough, in the form of a piecewise constant function. Now we consider another problem – we want to construct a control that is optimal in the sense of minimal energy (i.e. fuel) consumption during the movement, steering the beam from one assigned position of rest to another, given the time is greater than the minimal controllability time, that is $T > 4$. The idea of the framework is to observe that the optimal function lies in a closure of a specific linear span, which allows to rewrite the control as a solution of an appropriate moment problem. The obtained moment problem is non-Fourier – the eigenvalues of a connected operator are simple, but come in pairs asymptotically close to each other. It appears to be still possible to approximate such pairs of related functions – appearing in the moment moment equalities – corresponding to close eigenvalues by some divided differences. Besides, those functions are close to periodic ones, and this fact allows us to approximate the moment problem by another infinite moment problem of a special type. Such an idea was used for solving a problem of L_p optimal control of a string in [9]. In the end we obtain the finite set of linear equations and solve it numerically for some given data. The approach of replacing an infinite set of equations of some non-Fourier moment problem by a finite set is used, for example, in [8] & [28]. A similar idea for a non-homogeneous string was lately presented in [24].

Now we consider

Problem of optimal energy control: Let time $T > 0$ (large enough) and position $\theta_T \in \mathbb{R}$ be given. We steer the beam by angular acceleration of the disk, that is by $\ddot{\theta}$. For our convenience let us denote $u(t) = \ddot{\theta}(T-t)$ and assume that $T = 2M$, $M \in \mathbb{N}$, $M > 2$. We want to find a control $u \in L^2[0,T]$ such that it moves the beam from the position of rest at time $t = 0$ and angle $\theta = 0$, i.e. the position for which condition (5) holds, to the position of rest at time $t = T$ and angle θ_T, i.e.

$$\begin{array}{l} w(x,T) = \dot{w}(x,T) = 0, \\ \xi(x,T) = \dot{\xi}(x,T) = 0, \\ \dot{\theta}(T) = 0, \\ \theta(T) = \theta_T, \end{array} \tag{11}$$

$x \in [0,1]$. Moreover, we want u to be optimal in the following sense:

$$\int_0^T u^2(t)dt \to \min. \tag{12}$$

We mentioned above (see [15]) that if $T \geq 4$ the problem of finding a rest to rest control (1) & (11) is solvable, and the solution is a real function. It is proven that it is equivalent to solving the following moment problem

$$\begin{aligned} &\int_0^T e^{i\sqrt{\lambda_n}t} u(t)dt = 0, \quad n \in \mathbb{Z}, \\ &\int_0^T u(t)dt = 0, \\ &\int_0^T tu(t)dt = \theta_T, \end{aligned} \tag{13}$$

where for $n > 0$ λ_n is an eigenvalue of the operator connected with the beam equation (1), and for $n \leq 0$ we put $\lambda_{-n} = -\lambda_{n+1}$. In [16] an analysis of values λ_n is given, in particular it is shown that for $n > 0$ we have

$$\sqrt{\lambda_n} = \begin{cases} \frac{2k-1}{2}\pi - \varepsilon_n & \text{for} \quad n = 2k-1, \\ \frac{2k-1}{2}\pi + \varepsilon_n & \text{for} \quad n = 2k, \end{cases} \qquad \varepsilon_n = \mathcal{O}(\tfrac{1}{n}), \tag{14}$$

which coincides with (4) in our case. The moment problem (13) has a major disadvantage – its family of functions appearing in the moment problem contains two families of exponential functions with exponents that approach one another, therefore it does not constitute a Riesz basis. We will work this out by considering an equivalent moment problem. It is easy to see that each pair of moment equations from (13), corresponding to the eigenvalues close to each other, that is

$$\left.\begin{aligned} &\int_0^T e^{i\sqrt{\lambda_{2k-1}t}} u(t)dt = 0, \\ &\int_0^T e^{i\sqrt{\lambda_{2k}}t} u(t)dt = 0, \end{aligned}\right\}$$

is equivalent to a pair of equations

$$\left.\begin{aligned} &\int_0^T e^{i\sqrt{\lambda_{2k-1}t}} u(t)dt = 0, \\ &\int_0^T \frac{e^{i\sqrt{\lambda_{2k}t}} - e^{i\sqrt{\lambda_{2k-1}t}}}{\sqrt{\lambda_{2k}t} - \sqrt{\lambda_{2k-1}t}} u(t)dt = 0, \end{aligned}\right\}$$

which allows us to rewrite the original moment problem (13) as

$$\begin{aligned} &\int_0^T e^{i\sqrt{\lambda_k}t} u(t)dt = 0, \\ &\qquad -N+1 \leq 2k-1 < N, \\ &\int_0^T e^{i\sqrt{\lambda_{2k-1}}t} u(t)dt = 0, \\ &\qquad -N+1 > 2k-1 \text{ or } 2k-1 > N, \\ &\int_0^T \frac{e^{i\sqrt{\lambda_{2k}}t} - e^{i\sqrt{\lambda_{2k-1}}t}}{\sqrt{\lambda_{2k}}t - \sqrt{\lambda_{2k-1}}t} u(t)dt = 0, \\ &\qquad -N+1 > 2k-1 \text{ or } 2k-1 > N, \\ &\int_0^T u(t)dt = 0, \\ &\int_0^T tu(t)dt = \theta_T, \end{aligned} \tag{15}$$

for a fixed even integer N, with generating functions constituting a Riesz basis (see [19]).

It is worth to remind that we deal with a complicated set of generating vectors, they are not biorthogonal to each other. The classical general methods, like truncation or Galerkin approximation, do not use any properties of a moment problem generating vectors. In case of not biorthogonal generating vectors the resulting speed of convergence may be even less predictable. Here we propose another approach. The moment problem in question, (15), is generated by some special functions. We exploit the fact that the exponents $\sqrt{\lambda_n}$ have a special asymptotic behavior, given by (14). This means that those exponential functions are in fact close to trigonometric functions – at least for large indices. Knowing this we try not to truncate the problem, but to find the orthogonal complement subspace, and to seek for the approximation of the optimal solution in this subspace. The speed of convergence of obtained approximations is yet to be established, but one may expect it to be significantly better than of the one obtained by any general method.

After thorough analysis of asymptotic behavior of those functions, using some theorems of quadratically ε-close families of exponentials, one can see (cf. [24]) that the optimal solution of moment problem (15) can be approximated by a solution that is optimal in the sense of (12) of

$$\begin{aligned} &\int_0^T e^{i\sqrt{\lambda_n}t} u(t)dt = 0, \\ &\qquad \left\lfloor \tfrac{-N-2}{2} \right\rfloor \leq n \leq \left\lfloor \tfrac{N+1}{2} \right\rfloor, \\ &\int_0^T e^{i\frac{2n+1}{2}\pi t} u(t)dt = 0, \\ &\qquad \left\lfloor \tfrac{-N-2}{2} \right\rfloor \geq n \text{ or } n \geq \left\lfloor \tfrac{N+1}{2} \right\rfloor, \\ &\int_0^T te^{i\frac{2n+1}{2}\pi t} u(t)dt = 0, \\ &\qquad \left\lfloor \tfrac{-N-2}{2} \right\rfloor \geq n \text{ or } n \geq \left\lfloor \tfrac{N+1}{2} \right\rfloor, \\ &\int_0^T u(t)dt = 0, \\ &\int_0^T tu(t)dt = \theta_T, \end{aligned} \tag{16}$$

where $N \in 2\mathbb{N}$ is large enough. For simplicity later on we will write "$|n| \leq N$" instead of "$\left\lfloor \frac{-N-2}{2} \right\rfloor \leq n \leq \left\lfloor \frac{N+1}{2} \right\rfloor$" and "$|n| > N$" instead of "$\left\lfloor \frac{-N-2}{2} \right\rfloor \geq n$ or $n \geq \left\lfloor \frac{N+1}{2} \right\rfloor$".

For a fixed even integer N define (φ_k) as a sequence of functions generating the moment problem (15), that is $e^{i\sqrt{\lambda_k}t}(|k| \leq N)$, $e^{i\sqrt{\lambda_{2k-1}}t}(|k| > N)$, $\frac{e^{i\sqrt{\lambda_{2k}}t} - e^{i\sqrt{\lambda_{2k-1}}t}}{\sqrt{\lambda_{2k}}t - \sqrt{\lambda_{2k-1}}t}(|k| > N)$, 1 and t, with $\varphi_0(t) = t$, and (φ_k^N) as a sequence of functions generating the moment problem (16), i.e. $e^{i\sqrt{\lambda_k}t}$ $(|k| \leq N)$, $e^{i\frac{2k+1}{2}\pi t}(|k| > N)$, $te^{i\frac{2k+1}{2}\pi t}(|k| > N)$, 1 and t, with $\varphi_0^N(t) = t$. Let (ψ_k) and (ψ_k^N) be the families biorthogonal to (φ_k) and (φ_k^N), respectively.

The following holds.

Theorem 4. *(see [24]). Let*

$$u_N = \sum_k \langle u_N, \varphi_k^N \rangle \psi_k^N, \qquad u_0 = \sum_k \langle u_0, \varphi_k \rangle \psi_k,$$

where φ_k^N, ψ_k^N, φ_k and ψ_k are defined above. Then

$$\lim_{N\to\infty} \|u_N - u_0\| = 0.$$

In consequence, the sequence of optimal solutions of moment problems (16) *converges to the optimal solution of a moment problem* (13).

Now after establishing the fact that the original moment problem (13) can be approximated by another moment problem (16) we will try to find an equivalent formulation of the latter, but now in a form of a finite number of equations. This in the end will enable the numerical analysis of the problem of construction of the optimal control.

At first let us observe that the set $S_N = \Big\{1, t, e^{i\sqrt{\lambda_n}t}(|n| \le N),\ e^{i\frac{2n+1}{2}\pi t}(|n| > N),\ te^{i\frac{2n+1}{2}\pi t}(|n| > N)\Big\}$ is an $\mathcal{L}$-basis of $L^2[0,T] = L^2[0,2M]$, that is it is a Riesz basis in closure of its linear span $V = \overline{\text{Lin}S_N}$. Any control $u \in L^2[0,2M]$ can be written in the unique form $u = u_1 + u_2$, where $u_1 \in V$ and $u_2 \in V^\perp$. Since we have $\|u\|^2 = \|u_1\|^2 + \|u_2\|^2$ then due to the nature of our moment problem (16), which is generated by elements of S_N, we see that the solution u has the least norm (i.e. it fulfills (12)) if and only if $\|u_2\| = 0$, that is if and only if $u \in V$. This allows us to express the (unique) optimal solution u as

$$\begin{aligned} u(t) = {} & \textstyle\sum_{|n|\le N} \alpha_n e^{i\sqrt{\lambda_n}t} + \sum_{|n|>N} \beta_n e^{i\frac{2n+1}{2}\pi t} \\ & + \textstyle\sum_{|n|>N} \gamma_n t e^{i\frac{2n+1}{2}\pi t} + A + Bt, \end{aligned}$$

where $\alpha_n,\ \beta_n,\ \gamma_n,\ A,\ B \in \mathbb{C}$ are (unknown) constants. Further on we rewrite

$$\sum_{|n|>N} \beta_n e^{i\frac{2n+1}{2}\pi t} = e^{i\frac{\pi}{2}t}\zeta(t), \qquad \sum_{|n|>N} \gamma_n t e^{i\frac{2n+1}{2}\pi t} = e^{i\frac{\pi}{2}t} t\eta(t),$$

where $\zeta,\ \eta \in L^2[0,2M]$ are (unknown) functions periodic on $[0,2]$. Thus

$$u(t) = \sum_{|n|\le N} \alpha_n e^{i\sqrt{\lambda_n}t} + e^{i\frac{\pi}{2}t}\zeta(t) + e^{i\frac{\pi}{2}t}t\eta(t) + A + Bt. \tag{17}$$

Directly from the definitions of ζ, η we see that

$$\begin{aligned} &\int_0^2 \zeta(t)e^{in\pi t}dt = 0, \quad |n| \le N, \\ &\int_0^2 \eta(t)e^{in\pi t} = 0, \quad |n| \le N. \end{aligned} \tag{18}$$

Now we want to change our time interval from $[0,2M]$ to $[0,2]$, where selected summands of (17) are either periodic or close to periodic. To this end we define

$$\widehat{f}(s) = \sum_{k=0}^{M-1} f(s+2k), \quad s \in [0,2]$$

for any $f \in L^2[0,2M]$. Substituting (17) into second line of (16) and using (18) we obtain

$$\begin{aligned} 0 &= \int_0^{2M} e^{i\frac{2n+1}{2}\pi t}u(t)dt = \sum_{k=0}^{M-1}\int_{2k}^{2k+2} e^{i\frac{2n+1}{2}\pi t}u(t)dt \\ &= \int_0^2 e^{in\pi s}\Bigg[e^{i\frac{\pi}{2}s}M\Bigg(\frac{1}{M}\sum_{|n|\le N}\alpha_n\widehat{(-1)^k e^{i\sqrt{\lambda_n}s}} \\ &\quad + e^{i\frac{\pi}{2}s}\Big(\zeta(s) + s\eta(s) + (M-1)\eta(s)\Big) \\ &\quad + \widehat{(-1)^k}\Big(A + Bs + (M-1)B\Big)\Bigg)\Bigg]ds \end{aligned}$$

for all $|n| > N$, therefore we can rewrite the expression in the square brackets as $\sum_{|p|\le N} a_p e^{ip\pi s}$ with some unknown constants $a_p \in \mathbb{C}$, $|p| \le N$. Proceeding the same way with third line of (16) we express a similar term as $\sum_{|p|\le N} b_p e^{ip\pi s}$ with some unknown constants $b_p \in \mathbb{C}$, $|p| \le N$. Using those we can derive formulas for ζ and η, namely

$$\begin{aligned} \eta(s) = \frac{3}{M(M^2-1)}\Bigg(& -\sum_{|m|\le N}\alpha_m\widehat{(-1)^k s e^{i\sqrt{\lambda_m}s}}e^{\frac{-i\pi s}{2}} \\ & + (M+s-1)\textstyle\sum_{|m|\le N}\alpha_m\widehat{(-1)^k e^{i\sqrt{\lambda_m}s}}e^{\frac{-i\pi s}{2}} \\ & + Ae^{\frac{-i\pi}{2}}\left(-(-1)^{M-1}\left(M-\tfrac{1}{2}\right) + \tfrac{1}{2} + \tfrac{1-(-1)^M}{2}(M-1)\right) \\ & + Be^{\frac{-i\pi s}{2}}\Big(-\tfrac{1-(-1)^M}{2} - 2M(M-1)s - 4(-M+1)\tfrac{M}{2} \\ & + (M+s-1)\left(\tfrac{1-(-1)^M}{2}s + (-1)^{M-1}\left(M-\tfrac{1}{2}\right) - \tfrac{1}{2}\right)\Big) \\ & + \textstyle\sum_{|p|\le N} b_p e^{ip\pi s}e^{-i\pi s} \\ & - \textstyle\sum_{|p|\le N} a_p(M+s-1)e^{ip\pi s}e^{-is\pi}\Bigg), \end{aligned}$$

and

$$\begin{aligned} \zeta(s) = & -\tfrac{1}{M}\textstyle\sum_{|m|\le N}\widehat{(-1)^k e^{i\sqrt{\lambda_m}s}}e^{\frac{-is\pi}{2}} \\ & -\tfrac{1}{M}A\tfrac{1-(-1)^M}{2}e^{\frac{-is\pi}{2}} \\ & -\tfrac{1}{M}Be^{\frac{-is\pi}{2}}\left(\tfrac{1-(-1)^M}{2}s + (-1)^{M-1}\left(M-\tfrac{1}{2}\right) - \tfrac{1}{2}\right) \\ & + \tfrac{1}{M}\textstyle\sum_{|p|\le N} a_p e^{ip\pi s}e^{-is\pi} - (M+s-1)\eta(s). \end{aligned}$$

Now using all above and the remaining (first, fourth and fifth) lines of (16) we finally obtain $6N+2$ linear equations of the following form:

$$\begin{aligned} \int_0^2 e^{-in\pi s}\Bigg(& -\sum_{|m|\le N}\alpha_m\widehat{(-1)^k s e^{i\sqrt{\lambda_m}s}}e^{\frac{-i\pi s}{2}} \\ & + \textstyle\sum_{|m|\le N}\alpha_m\widehat{(-1)^k e^{i\sqrt{\lambda_m}s}}e^{\frac{-i\pi s}{2}}(M+s-1) \\ & + Ae^{\frac{-i\pi}{2}}\left(-(-1)^{M-1}\left(M-\tfrac{1}{2}\right) + \tfrac{1}{2} + \tfrac{1-(-1)^M}{2}(M-1)\right)\Big) \\ & + Be^{\frac{-i\pi s}{2}}\Big(-\tfrac{1-(-1)^M}{2} - 2M(M-1)s - 4(-M+1)\tfrac{M}{2} \\ & \quad + (M+s-1)\left(\tfrac{1-(-1)^M}{2}s + (-1)^{M-1}\left(M-\tfrac{1}{2}\right) - \tfrac{1}{2}\right)\Big) \\ & + \textstyle\sum_{|p|\le N} b_p e^{ip\pi s}e^{-i\pi s}\Bigg)ds = 0, \quad |n| \le N, \end{aligned}$$

$$\begin{aligned}\int_0^2 \Bigg(& -\frac{1}{M}\sum_{|m|\le N} \widehat{(-1)^k e^{i\sqrt{\lambda_m}s}}\, e^{\frac{-is\pi}{2}} - \frac{1}{M}A\frac{1-(-1)^M}{2}e^{\frac{-is\pi}{2}} \\ & -\tfrac{1}{M}Be^{\frac{-is\pi}{2}}\left(\tfrac{1-(-1)^M}{2}s + (-1)^{M-1}\left(M-\tfrac12\right)-\tfrac12\right) \\ & -(M+s-1)\eta(s) \\ & + \tfrac{1}{M}\textstyle\sum_{|p|\le N} a_p e^{ip\pi s}e^{-is\pi}\Bigg)\, ds = 0, \quad |n|\le N,\end{aligned}$$

$$\begin{aligned}\sum_{|m|\le N}\alpha_m \int_0^{2M} & e^{i(\sqrt{\lambda_m}+\sqrt{\lambda_n})t}dt + \int_0^{2M} e^{i\frac{\pi}{2}t}\zeta(t)e^{i\sqrt{\lambda_n}t}dt \\ & + \textstyle\int_0^{2M} e^{i\frac{\pi}{2}t}t\eta(t)e^{i\sqrt{\lambda_n}t}dt + A\int_0^{2M} e^{i\sqrt{\lambda_n}t}dt \\ & +B\textstyle\int_0^{2M} te^{i\sqrt{\lambda_n}t}dt = 0, \quad |n|\le N,\end{aligned}$$

$$\begin{aligned}\sum_{|m|\le N}\alpha_m \int_0^{2M} & e^{i\sqrt{\lambda_m}t}dt + \int_0^{2M} e^{i\frac{\pi}{2}t}\zeta(t)dt \\ & + \textstyle\int_0^{2M} e^{i\frac{\pi}{2}t}t\eta(t)dt + A2M + B2M^2 = 0,\end{aligned}$$

and

$$\begin{aligned}\sum_{|m|\le N}\alpha_m \int_0^{2M} & te^{i\sqrt{\lambda_m}t}dt + \int_0^{2M} e^{i\frac{\pi}{2}t}t\zeta(t)dt \\ & + \textstyle\int_0^{2M} e^{i\frac{\pi}{2}t}t^2\eta(t)dt \\ & +A2M^2 + B\frac{8}{3}M^3 = \theta_T.\end{aligned}$$

After a careful investigation (which we do not present here due to a sheer complication of its form in the general case) one can easily eliminate ζ and η from above equations and in the end obtain a Cramer system of $6N+2$ linear equations with exactly $6N+2$ unknown values, namely α_m $(|m| \le N)$, a_p $(|p| \le N)$, b_p $(|p| \le N)$, A and B.

This way we showed that solving the approximate optimal moment problem (16) can be reduced to solving a finite number of linear equations which in turn can be solved numerically. Note that entries of Cramer matrix of this system depend on the first N eigenvalues λ_n, which also have to be found numerically, but using the analytic formulas from [16] that can be done with any required level of precision. Thus we find the approximation of the infinite dimensional problem of optimal control (1) & (12) as a solution of a system of a finite number of linear equations.

One should notice that the method presented herein uses only one approximation step – replacing the moment problem (15) by a close one (16) – and the latter can be solved without any further discretization error.

The appropriate numerical illustration of the method was made for $T = 12$, $\theta_T = \frac{\pi}{2}$, $N = 4$, using *Scientific WorkPlace* software from *MacKichan*. Figure 1 shows the resulting angular acceleration of the disk.

VI. Perspectives

The new numerical method obtained above may be used for determining the optimal control in the other PDE models, for example for non-homogeneous Timoshenko beam model described earlier. The authors plan to apply this method also for a Timoshenko beam attached to a beam rotating in a plane perpendicular to the beam (cf. [5]).

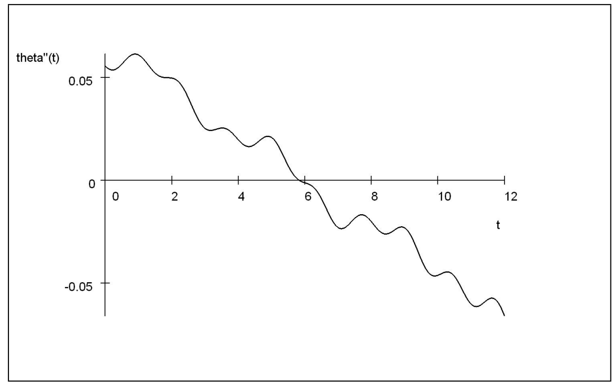

Fig. 1. Graph of optimal control $\ddot{\theta}(t) = u(T-t)$

References

[1] Avdonin S. A., Ivanov S. A.: *Families of Exponentials*, Cambridge Univ. Press, 1995.

[2] Avdonin S. A., Ivanov S. A.: *Exponential Riesz bases of subspaces and divided differences* (in Russian), St Petersburg Mathematical Journal, 13 (2001), No. 3, 339–351.

[3] Avdonin S. A., Moran W.: *Ingham type inequalities and Riesz bases of divided differences*, Int. J. Appl. Math. Comput. Sci., 11 (2001), no. 4, 101–118.

[4] Ballas K.: *Steuerung eines rotierenden, flexiblen Balkens mit einem Drehmoment von minimaler L^2 norm*, Diplomarbeit, TU Darmstadt 1994.

[5] Chentouf B., Wang J.-M.: *Stabilization and optimal decay rate for a non-homogeneous rotating body-beam with dynamic boundary controls*, J. Math. Anal. Appl. 318 (2006), 667–691.

[6] Delfour M. C., Kern M., Passeron L., Sevenne B.: *Modelling of a rotating flexible beam*, In: Control of Distributed Parameter Systems (ed.: H. E. Rauch), Pergamon Press, Los Angeles 1986, 383–387.

[7] Gohberg I.C., Krein M.G.: *Introduction to the theory of linear nonselfadjoint operators*, AMS Translation of Mathematical Monographs, vol. 18, AMS Providence 1969.

[8] Gröchenig K., Strohmer T.: *Numerical and theoretical aspects of nonuniform sampling of band-limited images*, in: Nonuniform Sampling, Theory and Practice, in: Inf. Technol. Transm. Process. Storage, Kluwer Aademic/Plenum Publishers, New York 2001, pp. 283–324.

[9] Gugat M., Leugering G., Sklyar G.M.: *Lp-optimal boundary control for the wave equation*, SIAM J. Control Optim. 44 (2005), 49–74.

[10] Krabs W.: *On Moment Theory and Controllability of One-Dimensional Vibrating Systems and Heating Processes* (Lecture Notes in Control and Information Sciences vol. 173), Springer-Verlag, Berlin et al. 1992.

[11] Krabs W.: *Controllability of a rotating beam*, In: Analysis and Optimization of Systems: State and Frequency Domain Approaches for Infinite-Dimesional Systems (Lecture Notes in Control and Information Sciences vol. 185, ed.: R. F. Curtain), Springer-Verlag, Berlin 1993, 447–458.

[12] Leugering G.: *Control and stabilization of a flexible arm*, Dynamics and Stability of Systems 5 (1990), 37–46.

[13] Leugering G.: *On control and stabilization of a rotating beam by applying moments at the base only*, In: Optimal Control of Partial Differential Equations (Lecture Notes in Control and Informational Sciences vol. 149, eds.: K. H. Hoffman and W. Krabs), Springer-Verlag, Berlin 1991, 182–191.

[14] Korobov V. I., Krabs W., Sklyar G. M.: *Construction of the control realizing the rotation of a Timoshenko beam*, J. Optim. Theory and Appl. 197 (2000), 51–68.

[15] Korobov V. I., Krabs W., Sklyar G. M.: *On the Solvability of Trigonometric Moment Problems Arising in the Problem of Controllability of Rotating Beams*, International Series of Numerical Mathematics 139 (2001), 145–156.

[16] Krabs W., Sklyar G. M.: *On the Controllability of a Slowly Rotating Timoshenko Beam*, Z. Anal. Anw. 18 (1999), 437–448.

[17] Krabs W., Sklyar G. M.: *On the Stabilizability of a Slowly Rotating Timoshenko Beam*, Z. Anal. Anw. 19 (2000), 131–145.

[18] Krabs W., Sklyar G. M.: *Controllability of Linear Vibrations*, NOVA Science Publishers Inc., Huntington, NY 2002, 163 pp.

[19] Krabs W., Sklyar G. M., Wozniak J.: *On the Set of Reachable States in the Problem of Controllability of Rotating Timoshenko Beams*, Journal for Analysis and its Applications 22 (2003), No. 1, 215–228.

[20] Russell D. L.: *Nonharmonic Fourier series in control theory of distributed parameter systems*, J. Math. Anal. Appl. 18 (1967), 542–560.

[21] Shubov M. A.: *Spectral operators generated by Timoshenko beam model*, System & Control Letters 38 (1999), 249–258.

[22] Sklyar G. M., Szkibiel G.: *Spectral properties of non-homogeneous Timoshenko beam and its rest to rest controllability*, J. Math. Anal. Appl. 338 (2008), 1054-1069.

[23] Sklyar G. M., Szkibiel G.: *Controllability from rest to arbitrary position of the nonhomogeneous Timoshenko beam*, Zh. Mat. Fiz. Anal. Geom. 4 (2008), 305-318.

[24] Sklyar G. M., Szkibiel G.: *Approximation of extremal solution of non-Fourier moment problem and optimal control for non-homogeneous vibrating systems*, J. Math. Anal. Appl. 387 (2012), 241–250.

[25] Sklyar G. M., Wozniak J.: *A Description of Smoothness of Reachable States in the Problem of Controllability of a Rotating Timoshenko Beam*, 8th IEEE Intern. Conf. MMAR, Szczecin 2002, 439–441.

[26] Sklyar G. M., Wozniak J.: *Exact Description of Controllable States in the Problem of Controllability of a Rotating Beam*, 10th IEEE Intern. Conf. MMAR, Miedzyzdroje 2004, 377–379.

[27] Sklyar G. M., Wozniak J.: *Ullrich conditions and smoothness of reachable states of a rotating beam*, J. Math. Anal. Appl. 354 (2009), 31-45.

[28] Strohmer T.: *Numerical analysis of the non-uniform sampling problem*, J. Comput. Appl. Math. 122 (2000) 297–316.

[29] Taylor S.W., Yau S.C.B.: *Boundary control of a rotating Timoshenko beam*, ANZIAM J. 2003, E143–E184.

[30] Ullrich D.: *Divided Differences and Systems of Nonharmonic Fourier Series*, Proc. Amer. Soc. 80 (1980), 47–57.

[31] Woittennek F., Rudolph J.: *Motion planning and boundary control for a rotating Timoshenko beam*, Proc. Appl. Math. Mech. 2 (2003), 106–107.

[32] Xiao-Jin Xiong: *Control and Computer Simulation of a Rotating Timoshenko Beam*, Ph.D. Thesis, McGill Univ., Montreal, 1997.