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(Generalized) Positive Rational Functions - a Structural Overview

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We here give a brief account of a long ongoing work.

Let $\mathbb{C}_+$ and $\mathbb{C}_-$ be the open right and left halves of the complex plane respectively, and let $\overline{\mathbb{P}}$ be the set of all positive semidefinite matrices. Recall that a $p \times p$-valued function $F(s)$, analytic in $\mathbb{C}_+$ is said to be *positive* if

$$F(s) + F(s)^* \in \overline{\mathbb{P}} \qquad s \in \mathbb{C}_+ \ .$$

The study of rational positive functions, denoted by $\mathcal{P}$, has been motivated from the 1920's by (lumped) electrical networks theory, see e.g. [5], [7] and [22]. From the 1960's positive functions also appeared in books on absolute stability theory, see e.g. [18], [19].

Physical systems are often real, so engineers typically refer to Positive Real functions. In this work, for convenience we adopt the complex framework.

A $p \times p$-valued function of bounded type in $\mathbb{C}_+$ (i.e. a quotient of two functions analytic and bounded in $\mathbb{C}_+$) is called *generalized positive* $\mathcal{GP}$ if

$$F(i\omega) + F(i\omega)^* \in \overline{\mathbb{P}} \quad a.e. \qquad \omega \in \mathbb{R},$$

where $F(i\omega)$ denotes the non-tangential limit of F at the point $i\omega$.

We here address several structural properties of these sets.

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- Even-Odd partitioning.

For arbitrary rational function $G(s)$ we denote,

$$G^{\#}(s) := G^{*}(-s^{*}).$$

Subsequently, the Even and Odd parts of $F(s)$ are

$$F_{\text{even}} = \tfrac{1}{2}(F + F^{\#}) \qquad\qquad F_{\text{odd}} = \tfrac{1}{2}(F - F^{\#}).$$

For example positive real odd functions are described by immittance of L-C circuits.

It turns out that $F \in \mathcal{GP}$ if and only if $F_{\text{even}} \in \mathcal{GP}$. Now, Even $\mathcal{GP}$ ($\mathcal{GPE}$) functions are those which admit (pseudo) spectral factorization, see e.g. [16], [20] and [21]

- Factorization of generalized positive functions.

It has been shown, in various frameworks see e.g. [12], [14], [15] and [17] that

$$F \in \mathcal{GP} \quad \Longleftrightarrow \quad F = GPG^{\#}, \quad P \in \mathcal{P}$$

where $G^{\#}(s)$ is as above.
In particular, if function $P(s)$ is in the middle is of a zero degree, this is the classical (pseudo) spectral factorization.

- The above sets $\mathcal{P}$, $\mathcal{GP}$, and their Even or Odd parts are closed under the following operations:

  (i) Positive scaling,

  (ii) Summation,

  (iii) Inversion,

  (iv) Composition,

As an illustration, common use of these operations are: (i) Dynamic output feedback or (ii) Series/parallel connection of immittances.

Thus this seemingly simple structural observation leads to both practical and theoretical open problems.

- State space realization.

Let $F(s)$ be a $p \times p$-valued rational function. Whenever $\lim_{s \to \infty} F(s)$ exists, this rational function is said to admit state-space realization, i.e.

$$F(s) = C(sI_n - A)^{-1}B + D \qquad R := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

where R is a $(n+p) \times (n+p)$ realization matrix.

We will review the fact that state space realizations of $\mathcal{P}$, $\mathcal{GP}$, Even and Odd parts can be characterized through a corresponding Kalman-Yakubovich-Popov Lemma, see e.g. [4], [6], [13].

This leads to the introduction of study of properties of *families* of realization matrices.

As an illustration, if $\phi(s)$ is a scalar $\mathcal{P}$-Odd rational function and R is a minimal realization matrix of $\mathcal{P}/\mathcal{P}$-Odd/$\mathcal{GPE}$ etc. matrix-valued ration function, then $\phi(R)$ is a realization matrix of a matrix-valued rational function of the same: Dimensions, Class while the McMillan degree does not increase.

Even a simple operation like R^{-1} leads to unusual conclusions.

In principle, one needs a whole presentation for each of the above four items (•). Hence, we shall only briefly touch upon them.

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