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From Tensor to Coupled Matrix/Tensor Decomposition*

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***Abstract*— Decompositions of higher-order tensors are becoming more and more important in signal processing, data analysis, machine learning, scientific computing, optimization and many other fields. A new trend is the study of coupled matrix/tensor decompositions (i.e., decompositions of multiple matrices and/or tensors that are linked in one or several ways). Applications can be found in various fields and include recommender systems, advanced array processing systems, multimodal biomedical data analysis and data completion. We give a short overview and discuss the state-of-the-art in the generalization of results for tensor decompositions to coupled matrix/tensor decompositions. We briefly discuss the remarkable uniqueness properties, which make these decompositions important tools for signal separation. Factor properties (such as orthogonality and triangularity, but also nonnegativity, exponential structure, etc.) may be imposed when useful but are not required for uniqueness per se. Also remarkable, in the exact case the decompositions may under mild conditions be computed using only tools from standard linear algebra. We touch upon the computation of inexact decompositions via numerical optimization. We illustrate some of the ideas using Tensorlab, a Matlab toolbox for tensors and tensor computations that we have recently released, and of which version 2 provides a comprehensive framework for the computation of (possibly constrained) coupled matrix/tensor decompositions.**

I. TENSOR DECOMPOSITIONS

A *canonical polyadic decomposition* of a rank-R tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ is a decomposition in a linear combination of R rank-1 terms [1], [12]:

$$\mathcal{A} = \sum_{r=1}^{R} \mathbf{u}_r^{(1)} \circ \mathbf{u}_r^{(2)} \circ \mathbf{u}_r^{(3)}, \tag{1}$$

i.e.,

$$a_{i_1 i_2 i_3} = \sum_{r=1}^{R} u_{i_1 r}^{(1)} u_{i_2 r}^{(2)} u_{i_3 r}^{(3)}. \tag{2}$$

A *Tucker decomposition* of a multilinear rank-(R_1, R_2, R_3) tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ is a decomposition of the form

$$\mathcal{A} = \mathcal{S} \cdot_1 \mathbf{U}^{(1)} \cdot_2 \mathbf{U}^{(2)} \cdot_3 \mathbf{U}^{(3)}, \tag{3}$$

i.e.,

$$a_{i_1 i_2 i_3} = \sum_{r_1=1}^{R_1} \sum_{r_2=1}^{R_2} \sum_{r_3=1}^{R_3} s_{r_1 r_2 r_3} u_{i_1 r_1}^{(1)} u_{i_2 r_2}^{(2)} u_{i_3 r_3}^{(3)}, \tag{4}$$

where the matrices $\mathbf{U}^{(1)} \in \mathbb{R}^{I_1 \times R_1}$, $\mathbf{U}^{(2)} \in \mathbb{R}^{I_2 \times R_2}$ and $\mathbf{U}^{(3)} \in \mathbb{R}^{I_3 \times R_3}$ have full column rank and where the tensor $\mathcal{S} \in \mathbb{R}^{R_1 \times R_2 \times R_3}$ [26], [27], [13], [2], [3].

A *block term decomposition* of a tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times I_3}$ is a decomposition of $\mathcal{A}$ of the form

$$\mathcal{A} = \sum_{r=1}^{R} \mathcal{S}_r \cdot_1 \mathbf{U}_r^{(1)} \cdot_2 \mathbf{U}_r^{(2)} \cdot_3 \mathbf{U}_r^{(3)}, \tag{5}$$

where the matrices $\mathbf{U}_r^{(1)} \in \mathbb{R}^{I_1 \times R_{1r}}$, $\mathbf{U}_r^{(2)} \in \mathbb{R}^{I_2 \times R_{2r}}$, $\mathbf{U}_r^{(3)} \in \mathbb{R}^{I_3 \times R_{3r}}$ have full column rank and where the tensors $\mathcal{S}_r \in \mathbb{R}^{R_{1r} \times R_{2r} \times R_{3r}}$ have full multilinear rank [5], [6], [7].

II. COUPLED MATRIX/TENSOR DECOMPOSITIONS

Let a number of matrices and/or higher-order tensors be given. We assume that they can be decomposed as in (1), (3) or (5). We study the situation where these decompositions are coupled in the sense that each of them involves at least one factor that is shared with at least one of the other decompositions. This can be relaxed to the situation where factors only have generating variables in common.

We study the uniqueness of such coupled decompositions [21], [25], generalizing results from [14], [11], [9].

In several cases coupled decompositions can be computed exactly by means of standard linear algebra (i.e., essentially by solving (overdetermined) sets of linear equations and by computing (generalized) EVD) [22]. Here we generalize results obtained in [4], [10].

Uniqueness conditions and conditions for analytic computability can in some cases be alleviated by constraining factor matrices to be Vandermonde or orthogonal. We generalize results from [19], [20].

We discuss numerical optimization based computation [17], generalizing results from [15], [16]. Our new algorithms are available in Tensorlab v2.0 [18].

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We discuss applications in signal processing [23], [24], [25].

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