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A maximal entropy solution for a rational Leech problem

A.E. Frazho[1], S. ter Horst[2], and M.A. Kaashoek[3]

***Abstract*— For the strictly positive case the maximum entropy solution X to the Leech problem $G(z)X(z) = K(z)$ and $\|X\|_\infty = \sup_{|z|\leq 1} \|X(z)\| \leq 1$, with G and K stable rational matrix functions, is proved to be a stable rational matrix function. An explicit state space realization for X is given, and $\|X\|_\infty$ turns out to be strictly less than one. The matrices involved in this realization are computed from the matrices appearing in a state space realization of the data functions G and K. A formula for the entropy of X is also given.**

I. Main results

Let G and K be matrix-valued H^∞ functions on the open unit disc $\mathbb{D}$ of sizes $m \times p$ and $m \times q$, respectively, and let T_G and T_K denote the corresponding block lower triangular Toeplitz operators. A $p \times q$ matrix-valued H^∞ function X is called a *solution to the Leech problem associated with* G *and* K whenever

$$\begin{aligned} G(z)X(z) = K(z) \quad (z \in \mathbb{D}) \\ \|X\|_\infty = \sup_{z\in\mathbb{D}} \|X(z)\| \leq 1. \end{aligned}$$

In an unpublished note [17] Leech proved that the problem is solvable if and only if the operator $T_GT_G^* - T_KT_K^*$ is nonnegative. Later the Leech theorem was derived as a corollary of more general results; see, e.g., [18, page 107], [8, Section VIII.6]), and [2, Section 4.7].

Now assume in addition that G and K are rational. In that case, if the Leech problem associated with G and K is solvable, one expects the problem to have a stable rational matrix solution as well. However, a priori this is not clear, and the existence of rational solutions was proved only recently in [19] by reducing the problem to polynomials, in [16] by adapting the lurking isometry method used in [3], and in [11] by using a state space approach. For another perspective see [20], [21] and their references.

In the present paper G and K are also stable rational matrix functions. We assume additionally that the operator $T_GT_G^* - T_KT_K^*$ is strictly positive. It is then known from commutant lifting theory that the Leech problem has a unique maximum entropy solution, that is, the (unique) solution X to the Leech problem associated with G and K for which the quantity

$$\mathcal{E}(X) = \frac{1}{2\pi}\int_0^{2\pi} \ln\det[I_q - X(e^{\imath\omega})^* X(e^{\imath\omega})]d\omega \qquad (1)$$

is maximal.

To present a maximal entropy solution for the Leech problem, consider a minimal realization for $\begin{bmatrix} G(z) & K(z)\end{bmatrix}$ of the form:

$$\begin{bmatrix} G & K\end{bmatrix} = \begin{bmatrix} D_1 & D_2\end{bmatrix} + zC(I_n - zA)^{-1}\begin{bmatrix} B_1 & B_2\end{bmatrix}. \qquad (2)$$

Because the realization (2) is minimal, the matrix A is stable and the observability operator W_{obs},

$$W_{obs} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} : \mathbb{C}^n \to \ell^2_+(\mathbb{C}^m), \qquad (3)$$

is one-to-one.

As a first step towards our main result we first derive, in Theorem 1.1 below, a necessary and sufficient condition for $T_GT_G^* - T_KT_K^*$ to be strictly positive in terms of the matrices in (2) and related matrices. To do this we need the rational $m \times m$ matrix function

$$R(z) = G(z)G^*(z) - K(z)K^*(z). \qquad (4)$$

Here $G^*(z) = G(\bar{z}^{-1})^*$ and $K^*(z) = K(\bar{z}^{-1})^*$. By T_R we denote the Toeplitz operator defined by R. Using the realization (2) one shows (see [11, Lemma 3.1]) that R admits the following state space representation:

$$R(z) = zC(I - zA)^{-1}\Gamma + R_0 + \Gamma^*(zI - A^*)^{-1}C^*. \qquad (5)$$

Here R_0 and Γ are the matrices defined by

$$R_0 = D_1D_1^* - D_2D_2^* + C(P_1 - P_2)C^*, \qquad (6)$$

$$\Gamma = B_1D_1^* - B_2D_2^* + A(P_1 - P_2)C^*, \qquad (7)$$

and P_1 and P_2 are the unique solutions of the Stein equations:

$$P_1 - AP_1A^* = B_1B_1^* \quad \text{and} \quad P_2 - AP_2A^* = B_2B_2^*. \qquad (8)$$

Since A is stable, the above equations are solvable and the solutions are unique. Finally, we associate with R the algebraic Riccati equation:

$$Q = A^*QA + (C - \Gamma^*QA)^*(R_0 - \Gamma^*Q\Gamma)^{-1}(C - \Gamma^*QA).$$

We are now ready to state our main results.

Theorem 1.1: Assume that $\begin{bmatrix} G & K\end{bmatrix}$ is given by the minimal realization (2). Then the operator $T_GT_G^* - T_KT_K^*$ is

[1]A.E. Frazho is with the Department of Aeronautics and Astronautics, Purdue University, West Lafayette, IN 47907, USA, frazho@ecn.purdue.edu

[2]S. ter Horst is with the Unit for BMI, North-West University, Private Bag X6001-209, Potchefstroom 2520, South Africa, sanne.terhorst@nwu.ac.za

[3]M.A. Kaashoek is with the Department of Mathematics, VU University Amsterdam, De Boelelaan 1081a, NL-1081 HV Amsterdam, The Netherlands, m.a.kaashoek@vu.nl

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strictly positive if and only if the following two conditions hold.

(i) There exists a strictly positive matrix Q such that
  (a) $R_0 - \Gamma^* Q \Gamma$ is strictly positive,
  (b) Q satisfies the Riccati equation,
  (c) the matrix $A_0 = A - \Gamma(R_0 - \Gamma^* Q\Gamma)^{-1}(C - \Gamma^* QA)$ is stable.

(ii) The operator $Q^{-1} + P_2 - P_1$ is strictly positive.

In this case, the Toeplitz operator T_R is strictly positive and the inverse of the operator $T_G T_G^* - T_K T_K^*$ is given by

$$\left(T_G T_G^* - T_K T_K^*\right)^{-1} = T_R^{-1} + T_R^{-1} W_{obs} \Omega W_{obs}^* T_R^{-1}$$
$$\Omega = (P_1 - P_2)\big(I + Q(P_2 - P_1)\big)^{-1}.$$

The second main result shows that the maximum entropy solution is rational and provides a state space realization for this solution.

Theorem 1.2: Let G and K be stable rational matrix functions, and assume that $\begin{bmatrix} G & K \end{bmatrix}$ is given by the minimal realization (2). Furthermore, assume that $T_G T_G^* - T_K T_K^*$ is strictly positive, or equivalently, that items (i) and (ii) of Theorem 1.1 hold. Then the maximal entropy solution X to the Leech problem for G and K is a stable rational matrix function which is given by the following state space realization:

$$X = D_U D_V^{-1} + z\left(C_1 - D_U D_V^{-1} C_2\right)(I - zA^{\times})^{-1} B_0 D_V^{-1}.$$

Here, using the matrices appearing in Theorem 1.1, the matrices are defined by

$$\begin{aligned}
\Delta &= R_0 - \Gamma^* Q \Gamma \\
C_0 &= \Delta^{-1}(C - \Gamma^* QA) \\
A_0 &= A - \Gamma C_0; \\
C_j &= D_j^* C_0 + B_j^* Q A_0, \quad (j = 1, 2); \\
D_0 &= \Delta^{-1}(D_2 - \Gamma^* Q B_2) + C_0 \Omega C_2^*; \\
B_0 &= B_2 - \Gamma \Delta^{-1}(D_2 - \Gamma^* Q B_2) + A_0 \Omega C_2^*; \\
D_U &= D_1^* D_0 + B_1^* Q B_0 \\
D_V &= I_q + D_2^* D_0 + B_2^* Q B_0; \\
A^{\times} &= A_0 - B_0 D_V^{-1} C_2.
\end{aligned}$$

Moreover, the state matrix $A^{\times}$ is stable, the matrix D_V is strictly positive, and the entropy of X is given by

$$\mathcal{E}(X) = -\ln\det[D_V]. \qquad (9)$$

The norm $\|X\|_\infty = \sup_{|z|\leq 1} \|X(z)\|$ is strictly less than one, and the McMillan degree of X is less than or equal to the McMillan degree of $\begin{bmatrix} G & K \end{bmatrix}$.

The proofs of the previous two results based on the central formula for the commutant lifting theorem will be given elsewhere.

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