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FIR Stabilization in Discrete One-Sided Model-Matching Problems

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Abstract— **This note studies stabilization problems arising in discrete estimation (or preview tracking) problems when unstable exosystems are used to impose steady-state behavior constraints. The main theme is stabilizability conditions under FIR (finite impulse response) constraints. Necessary and sufficient stabilizability conditions are derived and all solutions are parametrized. Unlike the continuous-time case, where the problem is solvable under the standard detectability condition, stabilizability in the discrete-time case depends on the length of the impulse response and, sometimes, also on the smoothing lag (length of preview).**

Index Terms— **FIR systems, stabilization, model matching, polynomial interpolation**

I. Introduction and Problem Formulation

This note studies the problem of stabilizing the discrete system

$$G_e(z) = G_1(z) - K(z)G_2(z) \tag{1}$$

by a stable K, where G_1 and G_2 are given $p_1 \times m$ and $p_2 \times m$ rational transfer functions. Such stabilization problems arise in numerous estimation and, by duality, open-loop tracking problems as a way to impose steady-state behavior constraints via unstable exosystems [1], [2]. Stabilization here requires that K cancels all unstable modes of G_1 using unstable modes of G_2. We assume that G_1 and G_2 are given in terms of their common *minimal* realization

$$\begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline C_1 & D_1 \\ C_2 & D_2 \end{array}\right]. \tag{2}$$

In this case stability can be understood as the absence of unstable (outside of the open unit disk) poles in the transfer function $G_e(z)$.

If K is any stable IIR (infinite impulse response) system, then, naturally, G_e is stabilizable iff the pair (C_2, A) is detectable [3, Lemma 3.6]. In this paper we consider a more restrictive setting, in which K is constrained to be FIR (finite impulse response) of the form

$$K(z) = \sum_{i=-\varkappa}^{\lambda-\varkappa-1} K_i z^{-i}, \tag{3}$$

where $\lambda \in \mathbb{N}$ is its length and $\varkappa \in \mathbb{Z}_{0..\lambda-1}$ is the smoothing lag, i.e., the number of future measurements, and $K_i \in \mathbb{R}^{p_1 \times p_2}$. FIR solutions might be advantageous in some estimation applications, for example to reduce sensitivity to temporary changes in signals or model parameters [4]. In the continuous-time case, FIR constraints change nothing from the stabilizability point of view, the stabilizability is guaranteed by the detectability of (C_2, A), see [5]. This changes in the discrete case. To see this, consider a simple example with $G_1(z) = 1/(z-1)$ and $G_2(z) = 1/(z-1)^2$. Stabilizability here reads [6, Ch. 10] as the existence of $K(z)$ satisfying the interpolation constraints $K(1) = 0$ and $K'(1) = 1$. If $K(z)$ is constrained to be static (FIR with $\lambda = 1$), then these constraints are obviously contradictory (because in the static case $K'(z) \equiv 0$). At the same time, the problem is solvable for $\lambda = 2$ by either $K(z) = 1 - z^{-1}$ (if $\varkappa = 0$) or $K(z) = z - 1$ (if $\varkappa = 1$).

The example above highlights another interpretation of the considered stabilization problem. Namely, it is essentially a polynomial interpolation problem [7]. We, however, are motivated by stabilization and hence prefer to solve the problem directly in terms of the realization in (2), without extracting interpolation constraints explicitly and without the need to account for multiple poles and directions. To the best of our knowledge, this approach has not been reported in the literature yet.

In this note we derive necessary and sufficient conditions for the existence of FIR K stabilizing (1). Unsurprisingly, the result reduces to the detectability of (C_2, A) if λ is sufficiently large. This is no longer true for shorter lengths of K, in which case we show that the stabilizability of G_e depends on C_1 and might even depend on the smoothing lag $\varkappa$. We also characterize all stabilizing K's when the problem is solvable.

II. Main Result

To formulate the main result of this note, let $\Pi_{\bar{s}}$ be any matrix such that $\operatorname{Im}\Pi_{\bar{s}}$ coincides with the unstable (in $|z| \geq 1$) spectral subspace of A and denote

$$\grave{C}_\lambda := \begin{bmatrix} C_2 \\ C_2 A \\ \vdots \\ C_2 A^{\lambda-1} \end{bmatrix}.$$

Then we have:

Theorem 1: Let G_1 and G_2 be given by their minimal realization (2). Then K of the form (3), which stabilizes G_e in (1), exists iff

$$\ker \grave{C}_\lambda \Pi_{\bar{s}} \subseteq \ker C_1 A^{\lambda-\varkappa-1} \Pi_{\bar{s}}. \tag{4}$$

Furthermore, the set of all FIR K's stabilizing G_e coincides then with the set of all solutions $\acute{K}_{\lambda,\varkappa}$ of

$$\acute{K}_{\lambda,\varkappa} \grave{C}_\lambda \Pi_{\bar{s}} = C_1 A^{\lambda-\varkappa-1} \Pi_{\bar{s}} \tag{5}$$

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with $\begin{bmatrix} K_{\lambda-\varkappa-1} & K_{\lambda-\varkappa-2} & \cdots & K_{-\varkappa} \end{bmatrix} = \acute{K}_{\lambda,\varkappa}$.

Before presenting the proof of Theorem 1, some remarks are in order.

Remark 1: Because the realization in (2) is assumed to be minimal, the detectability of (C_2, A) is necessary for (4) to hold true. Moreover, whenever λ is greater than or equal to the observability index of the unstable part of (C_2, A), $\ker \grave{C}_\lambda \Pi_{\bar{s}} = \ker \Pi_{\bar{s}}$ and hence (4) holds for every C_1 and $\varkappa$. In other words, if λ is sufficiently large (in the worst case, if $\lambda \geq \operatorname{rank} \Pi_{\bar{s}}$), then the stabilizability conditions reduce to the detectability of (C_2, A), exactly as in the IIR case.

Remark 2: The assumption that the realization in (2) is minimal can be made without loss of generality. Indeed, any hidden mode of (2) is also a hidden mode of G_e and hence does not affect its stability (understood as the absence of poles of the transfer matrix $G_e(z)$ in $|z| \geq 1$). Still, Theorem 1 can be easily adjusted to the nonminimal case. We actually only need to redefine $\Pi_{\bar{s}}$ as a matrix such that $\operatorname{Im} \Pi_{\bar{s}}$ coincides with the *controllable* unstable spectral subspace of A. Note that unobservable modes of $(\left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right], A)$ do not affect (4), so the detectability of (C_2, A) may then be relaxed.

Remark 3: For small λ, it might happen that (4) holds for some $\varkappa$ and does not hold for some other (see the example in Section III). In other words, the existence of a stabilizing FIR K might also depend on the smoothing lag (preview length) $\varkappa$. This never happens in the IIR case.

A. Proof of Theorem 1

Rewrite

$$K(z) = \acute{K}_{\lambda,\varkappa} \grave{S}(z)\, z^{1+\varkappa-\lambda},$$

where $\grave{S}(z) := \begin{bmatrix} I & zI & \cdots & z^{\lambda-1} I \end{bmatrix}'$ and $\acute{K}_{\lambda,\varkappa}$ is defined at the end of Theorem 1. Also, it follows by the equality

$$z^i (zI - A)^{-1} = A^i (zI - A)^{-1} + A^{i-1} + zA^{i-2} + \cdots + z^{i-1} I$$

that

$$G_1(z) z^{\lambda-\varkappa-1} = C_1 A^{\lambda-\varkappa-1} (zI - A)^{-1} B + D_\varkappa(z)$$

and

$$\grave{S}(z) G_2(z) = \grave{C}_\lambda (zI - A)^{-1} B + \grave{D}(z),$$

where $D_\varkappa(z)$ and $\grave{D}(z)$ are some matrix *polynomials* in z, whose exact form is irrelevant for the rest of the proof. Therefore,

$$G_e(z) = C_e (zI - A)^{-1} B\, z^{1+\varkappa-\lambda} + (D_\varkappa(z) - \acute{K}_{\lambda,\varkappa} \grave{D}(z))\, z^{1+\varkappa-\lambda}, \quad (6)$$

where $C_e := C_1 A^{\lambda-\varkappa-1} - \acute{K}_{\lambda,\varkappa} \grave{C}_\lambda$. As the second term in the right-hand side of (6) is stable (it is FIR) and the term $z^{1+\varkappa-\lambda}$ does not affect stability, the stability of G_e is equivalent to that of $C_e (zI - A)^{-1} B$. Because (A, B) is controllable (the realization in (2) is minimal), the latter system is stable iff all unstable modes of A are not observable through C_e. This condition reads $C_e \Pi_{\bar{s}} = 0$. Thus, the stabilizability of G_e is equivalent to the solvability of $C_e \Pi_{\bar{s}} = 0$ in $\acute{K}_{\lambda,\varkappa}$, which, in turn, is equivalent to (4). The characterization of the set of all stabilizing K's is then straightforward. This completes the proof.

III. Example

Consider the stabilization problem for

$$\begin{bmatrix} G_1(z) \\ G_2(z) \end{bmatrix} = \begin{bmatrix} z^2 + \alpha_1 z + \alpha_0 \\ z^2 \end{bmatrix} \frac{1}{(z-1)^2 (z+1)}$$
$$= \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ \hline \alpha_0 & \alpha_1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

In this case we can choose $\Pi_{\bar{s}} = I$.

Let $\lambda = 1$. Condition (4) then reads

$$\ker \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \subseteq \ker \begin{bmatrix} \alpha_0 & \alpha_1 & 1 \end{bmatrix}$$

and obviously holds iff $\alpha_0 = \alpha_1 = 0$. This corresponds to the case when $G_1 = G_2$ and is not surprising, taking into account that with a static $K(z) = K_0$,

$$G_e(z) = \frac{(1 - K_0) z^2 + \alpha_1 z + \alpha_0}{z^3 - z^2 - z + 1}$$

is either zero (iff $\alpha_0 = \alpha_1 = 1 - K_0 = 0$) or unstable because its two zeros are not sufficient to cancel all three unstable poles.

Now, let $\lambda = 2$. In this case

$$\ker \grave{C}_\lambda = \ker \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \operatorname{Im} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

If $\varkappa = 0$, (4) reads

$$\operatorname{Im} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \subseteq \ker \begin{bmatrix} -1 & 1 + \alpha_0 & 1 + \alpha_1 \end{bmatrix}$$

and holds true iff $\alpha_0 = 0$. The necessity of this condition can again be seen via the degree of the numerator of

$$G_e(z) = \frac{(1 - K_0) z^2 + (\alpha_1 - K_1) z + \alpha_0}{z^3 - z^2 - z + 1},$$

which is too low to cancel all three unstable poles. We therefore need to render $G_e(s) = 0$ and the condition $\alpha_0 = 0$ is the only possibility achieve that. To zero the other coefficients of the numerator, we need $K_1 = \alpha_1$ and $K_0 = 1$, which is indeed the unique solution of (5),

$$\begin{bmatrix} K_1 & K_0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 + \alpha_1 \end{bmatrix}$$

yielding $K(z) = 1 + \alpha_1 z^{-1}$. If $\varkappa = 1$, (4) reads

$$\operatorname{Im} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \subseteq \ker \begin{bmatrix} \alpha_0 & \alpha_1 & 1 \end{bmatrix}$$

and holds true iff $\alpha_0 + \alpha_1 = 0$. The necessity of this condition becomes apparent from

$$G_e(z) = \frac{-K_{-1} z^3 + (1 - K_0) z^2 + \alpha_1 z + \alpha_0}{z^3 - z^2 - z + 1},$$

whose numerator must be of the form $\alpha_0(z^3 - z^2 - z + 1)$ to cancel all poles. Then (5) reduces to

$$\begin{bmatrix} K_0 & K_{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \alpha_0 & -\alpha_0 & 1 \end{bmatrix}$$

and has a unique solution, $K_{-1} = -\alpha_0$ and $K_0 = 1 + \alpha_0$, which is what we need to render $G_e(z) = \alpha_0$. The stabilizing K is then $K(z) = -\alpha_0 z + 1 + \alpha_0$.

Thus, we have the following solvability conditions in terms of the parameters α_0 and α_1:

- if $\alpha_0 = \alpha_1 = 0$, the stabilization problem is solvable for all $\lambda \in \mathbb{N}$ irrespective of $\varkappa$;
- if $\alpha_0 = 0 \neq \alpha_1$, the problem is solvable for $\lambda = 2$ and $\varkappa = 0$ and for $\lambda \geq 3$ irrespective of $\varkappa$;
- if $\alpha_0 = -\alpha_1 \neq 0$, the problem is solvable for $\lambda = 2$ and $\varkappa = 1$ and for $\lambda \geq 3$ irrespective of $\varkappa$;
- otherwise, the problem is solvable iff $\lambda \geq 3$ (irrespective of $\varkappa$).

In the last three cases, the solution is unique iff $\lambda = 2, 3$. If $\lambda \geq 4$, there are infinitely many solutions. Consider, for instance, the case of $\lambda = 4$ and $\varkappa = 2$. By Theorem 1, K stabilizes G_e iff

$$\begin{bmatrix} 0 & -1 & -1 & -2 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_0 \\ K_{-1} \\ K_{-2} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 + \alpha_0 \\ 1 + \alpha_1 \end{bmatrix}$$

(mind the transpose of (5), taken to constrict the formulae). The set of all solutions to this equation is

$$\begin{bmatrix} K_1 \\ K_0 \\ K_{-1} \\ K_{-2} \end{bmatrix} = \begin{bmatrix} \alpha_0 + \alpha_1 \\ 1 + \alpha_0 \\ -\alpha_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} q,$$

where q is an arbitrary scalar. Then the set of all stable G_e's is $G_e(z) = \alpha_0 - qz$.

REFERENCES

[1] A. Saberi, A. A. Stoorvogel, and P. Sannuti, *Filtering Theory with Applications to Fault Detection, Isolation, and Estimation*. Boston: Birkhäuser, 2007.

[2] ——, *Control of Linear Systems with Regulation and Input Constraints*. London: Springer-Verlag, 2000.

[3] M. Kristalny, "Exploiting previewed information in estimation and control," Ph.D. dissertation, Faculty of Mechanical Eng., Technion—IIT, Aug 2010. [Online]. Available: http://library.technion.ac.il/thesis/mec/2317320.pdf

[4] W. H. Kwon, P. S. Kim, and P. Park, "A receding horizon Kalman FIR filter for discrete time-invariant systems," *IEEE Trans. Automat. Control*, vol. 44, no. 9, pp. 1787–1791, 1999.

[5] L. Mirkin and G. Tadmor, "Imposing FIR structure on H^2 preview tracking and smoothing solutions," *SIAM J. Control Optim.*, vol. 48, no. 4, pp. 2433–2460, 2009.

[6] W. Rudin, *Real and Complex Analysis*, 3rd ed. New York: McGraw-Hill, 1987.

[7] G. M. Phillips, *Interpolation and Approximation by Polynomials*. London: Springer-Verlag, 2003.