**21st International Symposium on**
**Mathematical Theory of Networks and Systems**
**July 7-11, 2014. Groningen, The Netherlands**

# HYPERBOLICITY FOR CURVES IN $\mathbb{P}^D$ AND THE GRASSMANIAN

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**Key words.** hyperbolic, determinantal representation, grassmannian

**EXTENDED ABSTRACT.** This talk is based on joint work with V. Vinnikov.

**Historical Background.** In this extended abstract we shall denote by $\mathbb{P}^d$ the complex projective $d$-dimensional space and by $\mathbb{P}^d(\mathbb{R})$ the real projective $d$-dimensional space. By $\mathbb{G}(\ell, d)$ we shall denote the Grassmannian of $\ell$-dimensional linear subspaces in $\mathbb{P}^d$. In fact $\mathbb{G}(\ell, d) = \mathrm{Gr}(\ell+1, d+1)$, the Grassmannian of $l+1$ dimensional spaces in $\mathbb{C}^{d+1}$.

The study of hyperbolic polynomials originated with partial differential equations. Linear hyperbolic partial differential equations are equations for which the Cauchy problem can be locally solved for non-characteristic hyperplanes. Lars Gärding published a pioneering work defining hyperbolic polynomials (and thus hyperbolic hyperplanes in $\mathbb{P}^d$). A homogeneous polynomial, $p \in \mathbb{R}[x_0, \ldots, x_d]$, is hyperbolic if there exists a real point $a$ such that $p(a) \neq 0$ and for every other real point $x$ the roots of $p(ta + x)$ are real. We will say that a point $a$ is a witness to the hyperbolicity of $p$. The cone over the set of all witnesses is called the hyperbolicity cone of $p$.

A closed set $\mathcal{C} \subset \mathbb{R}^{d+1}$ is called an algebraic interior of degree $d$ if there exists a polynomial of degree $m$ such that $\mathcal{C}$ is the closure of a connected component of $\{x \in \mathbb{R}^m \mid p(x) > 0\}$ and $m$ is the minimal degree among all polynomials that satisfy the above condition for $\mathcal{C}$.

In fact hyperbolicity cones of homogeneous polynomials provide the first examples of algebraic interiors. Another example is provided by LMI representable sets, i.e., sets $\mathcal{C}$ such that there exist symmetric real matrices $A_0, \ldots, A_d$, with $A_0$ positive definite and $\mathcal{C} = \overline{\{x \in \mathbb{R}^{d+1} \mid x_0 A_0 + \ldots + x_d A_d \geq 0\}}$. A natural question arises whether all algebraic interiors admit an LMI representation with size equal the degree of the algebraic interior. A positive answer was provided by J. W. Helton and V. Vinnikov based on earlier works of V. Vinnikov on determinantal representations. A negative answer for higher dimensions was provided, using different methods, by several authors. In particular using matroid theory P. Branden proved that even if one allows to take powers of your polynomial still there won't be any LMI representations available.

**Main Results.** In this work we chose to focus on a slightly different question, namely hyperbolicity and determinantal representability of subvarieties of codimension greater that 1. The idea originated from the fact that $\mathbb{P}^d = \mathrm{Gr}(1, d+1)$. If one considers $X \subset \mathbb{P}^d$ defined over the reals and of codimension $\ell$, then a natural notion of hyperbolicity arises. We say that $X$ is hyperbolic if there exists a real $\ell - 1$-dimensional plane $V \subset \mathbb{P}^d$, such that for every real $\ell$-dimensional plane $V \subset U \subset \mathbb{P}^d$, we have that $U$ intersects $X$ only at real points.

The incidence correspondence is the set $\Sigma \subset \mathbb{G}(\ell - 1, d) \times \mathbb{P}^d$, defined as follows:

$$\Sigma = \{(V, x) \mid x \in V\}.$$

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**ISBN: 978-90-367-6321-9** 

One notes that $\Sigma$ comes equipped with natural smooth projections both to $\mathbb{G}(\ell - 1, d)$ and t0 $\mathbb{P}^d$, $\pi_1$ and $\pi_2$, respectively. Define $Y = \pi_1(\pi_2^{-1}(X))$ and note that $Y$ is a hypersurface in the Grassmannian. It is easy to see now that the above notion of hyperbolicity corresponds to the following notion of hyperbolicity on the Grassamanian. A real hypersurface, $Y$, in $\mathbb{G}(\ell - 1, d)$ is hyperbolic if there exists a real point $V$, such that for every 1-dimensional real Schubert cycle, $S$, through $V$, $S$ intersects $Y$ only at real points.

We will deal with a specific type of determinantal representations. Given a tensor $\gamma \in \wedge^{k+1}\mathbb{C}^{d+1} \otimes M_n(\mathbb{C})$, we define the set.

$$D(\gamma) = \left\{\mu \in \mathbb{P}^d \mid \exists\ v \in \mathbb{C}^n \setminus 0\ ,(\gamma \wedge \mu)v = 0.\right\}.$$

Here we consider $\gamma \wedge \mu$ as a mapping from $\mathbb{C}^n$ to $\wedge^{k+2}\mathbb{C}^{d+1} \otimes C^n$. In other words $v$ is in the intersection of kernels of the coefficient of the basis vectors of $\wedge^{k+2}C^{d+1}$ in $\gamma \wedge \mu$. We will say that $\gamma$ is non degenerate if there exists a $d - k - 1$-dimensional plane in $\mathbb{P}^d$ that does not intersect $D(\gamma)$.

The set $D(\gamma)$ is in fact the degeneration locus of a vector bundle map on $\mathbb{P}^d$:

$$T(\gamma) = \gamma \wedge \mu \colon \mathcal{O}(-1)^n \to \mathcal{O}^n \otimes \wedge^{k+1}\mathcal{O}^d \cong \mathcal{O}^{n\binom{d+1}{k+1}}.$$

In particular $D(\gamma)$ is closed. We define the cycle associated to a non-degenerate $\gamma \in \wedge^{k+1}\mathbb{C}^{d+1} \otimes M_n(\mathbb{C})$ in $Z_*(\mathbb{P}^d)$ by:

$$Z(\gamma) = \sum_{j=1}^{r} n_j [D_j].$$

Here we denote by $D_j$ the irreducible components of $D(\gamma)$. The numbers $n_j$ are the dimensions of the generic fiber of the kernel of $T(\gamma)$ over $D_j$.

We say that $X$ admits a Livsic-type determinantal representation if $X = D(\gamma)$ as sets, for some non-degenerate $\gamma$.

We say that $\gamma$ is "nice" if $D(\gamma)$ is of pure dimension $k$. The following theorem is an easy generalization of a similar result in the classical theory:

THEOREM 0.1. *Assume that $X$ admits a "nice" symmetric Livsic-type determinantal representation and set $Y \subset \mathbb{G}(\ell - 1, d)$ to be the corresponding hypersurface. Then $Y$ admits a determinantal representation and furthermore, if the matrix defining $Y$ positive definite at some point, $V \in \mathbb{G}(\ell - 1, d)$, then $Y$ is hyperbolic with respect to $V$.*

The case of curves provides an interesting generalization of the classical theory in $\mathbb{P}^2$. Generalizing the methods of J. A. Ball and V. Vinnikov we are able to prove that every irreducible smooth projective curve admits a "nice" Livsic-type determinantal representation. Furthermore, we have that:

THEOREM 0.2. *If a curve $X \in \mathbb{P}^d$ is hyperbolic with respect to some $d - 2$-dimensional plane $V$, then the determinantal representation can be chosen so that $\gamma(V) = \gamma \wedge V$ is definite.*

Since the Grassmannian realized as a closed subvariety of $\mathbb{P}^N$ is projectively normal and $\mathrm{Pic}(\mathbb{G}(\ell - 1, d)) = \mathbb{Z}$ we get that every hypersurface $Y \subset \mathbb{G}(\ell - 1, d)$ is obtained via intersection with a hypersurface $Y'$ in the ambient space. In the case of curves one concludes that:

THEOREM 0.3. *In the above setting we have that there exists a determinantal representation for $Y'$ and $Y'$ is hyperbolic with respect to $V$.*

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