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Convergence Analysis of Multi-Agent Consensus with Noisy and Directed Communication

Ryosuke Morita[1], Takayuki Wada[2], Izumi Masubuchi[3], Toru Asai[4], and Yasumasa Fujisaki[2]

***Abstract*— In this paper, we analyze convergence of the consensus problems in multi-agent systems. The system considered here has a directed graph topology and the communication among the agents includes noise. A stochastic approximation method is applied to a consensus algorithm and the relation between the closeness of the agreement and the number of iterations is clarified. A required number of iterations for the desired closeness can be estimated by a probabilistic guarantee.**

***Index Terms*— Multi-agent system, Convergence analysis, Stochastic approximation**

I. INTRODUCTION

Multi-agent systems consists of a lot of partial systems called agents which exchange the information such as their states each other and achieve a global objective without any supervisor. To consider large scale network systems, it is not reasonable to take care of all of agents individually for the global objective. By taking such a system as a multi-agent system, we can control the whole system by only considering their own local objectives of the agents. Therefore, it is useful for these systems and various applications are expected [1]–[4].

Actually, influence of noise is not avoidable for multi-agent systems in most cases. Although analysis of the consensus problems with noisy communication among the agents have been also considered [5]–[8], the most of existing results are focused on the steady state. To apply multi-agent system theories to the real world, it needs further analysis which includes transient behavior.

The authors have analyzed the convergence of the multi-agent systems with undirected graph topology under noisy communication [9], where the relation between the closeness of the agreement and the number of iterations of the consensus algorithm has been clarified.

The presenting paper follows this line of research and we consider a directed graph case, that is, the network topology of the communication is asymmetric. In fact, agents need two kinds of structures such as senders and receivers for exchanging the information each other. Therefore, the communication among the agents is not symmetric in general. Note that the network topology is assumed to be a balanced graph for simplicity.

We here introduce the idea for a stochastic approximation method [10], [11] into a consensus problem for multi-agent systems. Then, we analyze the convergence of the consensus under noisy and directed communication. In particular, we provide the relation between the closeness of the agreement and the number of iterations of the consensus algorithm, while this kind of analysis has been explored in [12], [13] in the context of standard stochastic approximation.

II. PROBLEM FORMULATION

We consider a multi-agent system M including $N \in \mathbb{N}$ agents. The agents exchange the information along the network topology which is expressed by the balanced directed graph $G(V, E)$, where $V = \{1, 2, \dots, N\}$ is the set of the nodes and $E \subseteq V \times V$ is the set of edges. The set of the agent from which the agent i receives the information is denoted by $\mathcal{N}_i = \{j \in V | (i, j) \in E\}$.

If $j \in \mathcal{N}_i$, the agent i receives the information $y_{ij}(k) \in \mathbb{R}$, which is defined by

$$y_{ij}(k) = x_j(k) + w_{ij}(k)$$

where $k \in \mathbb{N}$ is the discrete time, $x_j(k) \in \mathbb{R}$ is the state of the agent j, and $w_{ij}(k) \in \mathbb{R}$ is the communication noise according to an independent and identically distribution with respect to i, j, and k. The communication noise $w_{ij}(k)$ also satisfies

$$\mathsf{E}[w_{ij}(k)] = 0, \qquad \mathsf{Var}[w_{ij}(k)] \le \sigma^2 < \infty.$$

For $j \notin \mathcal{N}_i$, it is defined as $w_{ij}(k) = 0$ because of no communication.

The agent i updates its state $x_i(k)$ by

$$x_i(k+1) = x_i(k) + r(k) \sum_{j \in \mathcal{N}_i} (y_{ij}(k) - x_i(k)) \qquad (1)$$

where $r(k) \in \mathbb{R}$ is the communication gain.

In the noiseless case, it is well-known that when the network topology is a balanced directed graph, the agents reach an agreement at the average of the initial values [1]. According to our previous result [9], this property also holds in the noisy case. To evaluate the closeness of the agreement,

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[1]Ryosuke Morita was with Graduate School of Information Science and Technology, Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan while this work was done. He is currently with College of Science and Engineering, Aoyama Gakuin University, 5-10-1 Fuchinobe, Chuo, Sagamihara, Kanagawa 252-5258, Japan r.morita.jp@ieee.org

[2]Takayuki Wada, and Yasumasa Fujisaki are with Graduate School of Information Science and Technology, Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan {t-wada,fujisaki}@ist.osaka-u.ac.jp

[2]Izumi Masubuchi is with Graduate School of System Informatics, Kobe University, 1-1 Rokkodai, Nada, Kobe 657-8501, Japan msb@harbor.kobe-u.ac.jp

[3]Toru Asai is with Graduate School of Engineering, Osaka University, 2-1 Yamadaoka, Suita, Osaka 565-0871, Japan tasai@mech.eng.osaka-u.ac.jp

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we introduce the average $\mu(k)$ of the states of the agents and the distance $\delta(k)$ from the average, which are given by

$$\mu(k) = \frac{1}{N}\sum_{i=1}^{N} x_i(k),$$
$$\delta(k) = [x_1(k)-\mu(k)\ x_2(k)-\mu(k)\ \cdots\ x_N(k)-\mu(k)]^\top.$$

Using these expression, the consensus problem here is to achieve

$$\lim_{k\to\infty}(x_i(k)-\mu(k)) = 0, \qquad \forall i,$$

by iterating the consensus algorithm (1). In addition, the closeness of the agreement can be quantitatively evaluated by $\|\delta(k)\|$. The purpose of this paper is clarify the relation between the closeness of the agreement and the number of iterations of the consensus algorithm for the multi-agent system with communication noise and a directed graph topology.

III. CLOSENESS OF THE AGREEMENT

The network topology of the system M is characterized by the graph Laplacian $L = [\ell_{ij}] \in \mathbb{R}^{N\times N}$, which is defined as

$$\ell_{ij} = \begin{cases} -1 & \text{if } (i,j)\in E, \\ d_i & \text{if } i=j, \\ 0 & \text{otherwise,} \end{cases}$$

where $d_i \in \mathbb{N}$ is the number of the elements of $\mathcal{N}_i$. Since $G(V,E)$ is a balanced directed graph, the graph Laplacian L satisfies

$$\mathbf{1}_N^\top L = 0, \qquad L\mathbf{1}_N = 0, \qquad \operatorname{rank} L = N-1,$$

where $\mathbf{1}_N \in \mathbb{R}^N$ is the vector whose elements are one, while it is usually $L \neq L^\top$.

Using L, the algorithm (1) can be written in

$$x(k+1) = (I_N - r(k)L)x(k) + r(k)W(k)\mathbf{1}_N, \quad (2)$$

where $x(k) = [x_1(k)\ x_2(k)\ \cdots\ x_N(k)]^\top$ and $I_N \in \mathbb{R}^{N\times N}$ is the identity matrix. The matrix $W(k) \in \mathbb{R}^{N\times N}$ consists of $w_{ij}(k)$, where its $\{i,j\}$-th element is $w_{ij}(k)$.

According to the definitions of $\mu(k)$ and $\delta(k)$, the algorithm can be rewritten as

$$\mu(k+1) = \mu(k) + r(k)\bar{w}(k), \quad (3)$$
$$\delta(k) = (I_N - r(k)L)\delta(k) + r(k)\tilde{w}(k), \quad (4)$$

where

$$\bar{w}(k) = \mathbf{1}_N^\top W(k)\mathbf{1}_N/N,$$
$$\tilde{w}(k) = (I_N - \mathbf{1}_N\mathbf{1}_N^\top/N)W(k)\mathbf{1}_N,$$

and these satisfy

$$\mathsf{E}[\bar{w}(k)] = 0, \quad \mathsf{E}[\tilde{w}(k)] = 0,$$
$$\mathsf{Var}[\bar{w}(k)] = \mathsf{E}[\bar{w}(k)^2] \le \sigma^2,$$
$$\mathsf{Cov}[\tilde{w}(k)] = \mathsf{E}[\tilde{w}(k)\tilde{w}(k)^\top] \preceq N\sigma^2(I_N - \mathbf{1}_N\mathbf{1}_N^\top/N).$$

Note that the update algorithms (3) and (4) are independent of each other.

Then, we furthermore employ a state coordinate transformation

$$\begin{bmatrix} \delta_a(k) \\ \delta_b(k) \end{bmatrix} = \begin{bmatrix} Z^\top \\ \mathbf{1}_N^\top/\sqrt{N} \end{bmatrix}\delta(k),$$
$$\delta(k) = \begin{bmatrix} Z & \mathbf{1}_N/\sqrt{N} \end{bmatrix}\begin{bmatrix} \delta_a(k) \\ \delta_b(k) \end{bmatrix}.$$

where

$$\begin{bmatrix} Z^\top \\ 1_N^\top/\sqrt{N} \end{bmatrix}\begin{bmatrix} Z & 1_N/\sqrt{N} \end{bmatrix} = \begin{bmatrix} Z & 1_N/\sqrt{N} \end{bmatrix}\begin{bmatrix} Z^\top \\ 1_N^\top/\sqrt{N} \end{bmatrix} = I_N.$$

By this transformation, $\delta_b(k)$ is always zero and $\|\delta_a(k)\| = \|\delta(k)\|$ holds. Therefore, we can evaluate the closeness of the agreement by only $\delta_a(k)$.

The update algorithm of $\delta_a(k)$ is

$$\delta_a(k+1) = (I_{N-1} - r(k)Z^\top LZ)\delta_a(k) + r(k)\tilde{w}_a(k),$$

where $\tilde{w}_a(k) = Z^\top\tilde{w}(k)$ and it satisfies

$$\mathsf{E}[\tilde{w}_a(k)] = 0,$$
$$\mathsf{Cov}[\tilde{w}_a(k)] = Z^\top\,\mathsf{Cov}[\tilde{w}(k)]Z \preceq N\sigma^2 I_{N-1}.$$

Since the graph $G(E,V)$ is a balanced graph, the following lemma holds.

Lemma 1: For the graph Laplacian L, there exist $\eta \in (0,1)$ and $\zeta \in (0,\infty)$ such that

$$\left\| I_{N-1} - \frac{Z^\top LZ}{\zeta} \right\| \le 1-\eta. \quad (5)$$

Proof: Note that $L + L^\top$ is a graph Laplacian of an undirected graph. Then, we see that $Z^\top(L+L^\top)Z$ is strictly positive definite matrix. Thus, there exists $\varepsilon > 0$ such that

$$Z^\top(L+L^\top)Z = Z^\top LZ + Z^\top L^\top Z \succeq \varepsilon I_{N-1} \succ 0 \quad (6)$$

In addition, $Z^\top L^\top LZ$ is a symmetric matrix so that there exists $\zeta \in (0,\infty)$ such that

$$\varepsilon I_{N-1} \succ \frac{Z^\top L^\top LZ}{\zeta} = \frac{(Z^\top LZ)^\top(Z^\top LZ)}{\zeta}.$$

For ζ, there also exists $\eta \in (0,1)$ such that

$$\frac{1}{\zeta}\left(\varepsilon I_{N-1} - \frac{(Z^\top LZ)^\top(Z^\top LZ)}{\zeta}\right) \succeq 2\eta I_{N-1} \succeq (2\eta-\eta^2)I_{N-1}. \quad (7)$$

Using (6) and (7), we have

$$\begin{aligned}
&(1-\eta)^2 I_{N-1} \\
&= I_{N-1} - (2\eta-\eta^2)I_{N-1} \\
&\succeq I_{N-1} - \frac{1}{\zeta}\left(\varepsilon I_{N-1} - \frac{(Z^\top LZ)^\top(Z^\top LZ)}{\zeta}\right) \\
&\succeq I_{N-1} - \frac{1}{\zeta}\left(Z^\top LZ + Z^\top LZ\right) + \frac{(Z^\top LZ)^\top(Z^\top LZ)}{\zeta^2} \\
&= \left(I_{N-1} - \frac{Z^\top LZ}{\zeta}\right)^\top\left(I_{N-1} - \frac{Z^\top LZ}{\zeta}\right).
\end{aligned}$$

Therefore, (5) is derived. ■

Now, suppose that $k_0 \in \mathbb{N}$ is chosen as

$$k_0 \geq \frac{\zeta}{2\eta} - 1$$

and select the communication gain as

$$r(k) = \frac{1}{2\eta\zeta(k_0 + k)}.$$

Then, the following theorem is derived for the convergence of a multi-agent consensus with noisy and directed communication. The proof is described in the next section.

Theorem 1: For given constants $\alpha \in (0, \infty)$, $\beta \in (0, \infty)$, and $\gamma \in (0, 1)$, select $k_f \in \mathbb{N}$ which satisfies

$$k_f \geq \max(\tau_1, \tau_2),$$
$$\tau_1 = \frac{k_0 + 1}{\alpha^2} - k_0, \quad \tau_2 = \frac{1}{k_0 + 1}\left(\frac{N(N-1)\sigma^2}{\beta^2\eta^2\zeta^2\gamma}\right) - k_0.$$

Then, for any initial state $x(1)$, the k_f-th variation $\delta(k_f)$ satisfies

$$\mathsf{P}\left(\|\delta(k_f)\| \leq \alpha\|\delta(1)\| + \beta\right) \geq 1 - \gamma,$$

where P is a probability on the measurement of $w_{ij}(k)$.

Theorem 1 gives the relation between the closeness of the agreement and the number of iterations of the consensus algorithm (1) in the sense of probability. The desired closeness of the agreement is specified by the parameters α and β, while the probabilistic guarantee is decided by γ.

Since the parameters α, β, and γ can be chosen as small values arbitrarily, we see that the state of the agents $x_i(k)$ converges in probability to the average of all the agents $\mu(k)$ theoretically. That is, we have the following corollary.

Corollary 1: For any initial state $x(1)$, the state of each agent $x_i(k)$ converges to the average $\mu(k)$ with probability one.

This means Theorem 1 contains the existing result as a special case.

IV. PROOF OF THE MAIN RESULT

To proof Theorem 1, we define the transition matrix $\Phi(p, q)$ as

$$\Phi(p, q) = (I_{N-1} - r(p)Z^\top LZ)(I_{N-1} - r(p-1)Z^\top LZ) \cdots (I_{N-1} - r(q)Z^\top LZ).$$

Using $\Phi(p, q)$, $\delta_a(k)$ is written in

$$\delta_a(k) = \Phi(k-1, 1)\delta_a(1) + \sum_{n=1}^{k-1} \Phi(k-1, n+1)r(n)\tilde{w}_a(n). \tag{8}$$

Thus, the expectation is

$$\mathsf{E}[\delta_a(k)] = \Phi(k-1, 1)\delta_a(1), \tag{9}$$

and the covariance is

$$\begin{aligned}\mathsf{Cov}[\delta_a(k)] &= \mathsf{E}[(\delta_a(k) - \mathsf{E}[\delta_a(k)])(\delta_a(k) - \mathsf{E}[\delta_a(k)])^\top] \\ &= \mathsf{E}\left[\left(\sum_{n=1}^{k-1}\Phi(k-1, n+1)r(n)\tilde{w}_a(n)\right)\right. \\ &\qquad \left.\left(\sum_{n=1}^{k-1}\Phi(k-1, n+1)r(n)\tilde{w}_a(n)\right)^\top\right] \\ &= \mathsf{E}\left[\sum_{n=1}^{k-1} r(n)^2\Phi(k-1, n+1)\tilde{w}_a(n)\right. \\ &\qquad \left.\tilde{w}_a(n)^\top\Phi(k-1, n+1)^\top\right]\end{aligned} \tag{10}$$

Here, we introduce $d(k) = \|\delta_a(k) - \mathsf{E}[\delta_a(k)]\|^2$. Then the expectation is

$$\begin{aligned}\mathsf{E}[d(k)] &= \mathsf{E}[(\delta_a(k) - \mathsf{E}[\delta_a(k)])^\top(\delta_a(k) - \mathsf{E}[\delta_a(k)])] \\ &= \mathsf{E}[\mathsf{Tr}\left((\delta_a(k) - \mathsf{E}[\delta_a(k)])(\delta_a(k) - \mathsf{E}[\delta_a(k)])^\top\right)] \\ &= \mathsf{Tr}\left(\mathsf{Cov}[\delta_a(k)]\right).\end{aligned} \tag{11}$$

By Markov's inequality, the following relation is satisfied:

$$\mathsf{P}\left(d(k) \leq \frac{\mathsf{Tr}\left(\mathsf{Cov}[\delta_a(k)]\right)}{\gamma}\right) \geq 1 - \gamma \tag{12}$$

For $\|\delta_a(k)\|$, it is clear that

$$\begin{aligned}\|\delta_a(k)\| &= \|\delta_a(k) - \mathsf{E}[\delta_a(k)] + \mathsf{E}[\delta_a(k)]\| \\ &\leq \|\delta_a(k) - \mathsf{E}[\delta_a(k)]\| + \|\mathsf{E}[\delta_a(k)]\|\end{aligned}$$

is satisfied. Applying this to (12), it is transformed into

$$\mathsf{P}\left(\|\delta_a(k)\| \leq \|\mathsf{E}[\delta_a(k)]\| + \sqrt{\frac{\mathsf{Tr}\left(\mathsf{Cov}[\delta_a(k)]\right)}{\gamma}}\right) \geq 1 - \gamma. \tag{13}$$

In the following part, we derive the guarantee of the upper bound.

From the inequality (5), we have

$$\begin{aligned}&\left\|I_{N-1} - \frac{Z^\top LZ}{2\eta\zeta(k_0 + k)}\right\| \\ &\leq \left\|I_{N-1} - \frac{I_{N-1}}{2\eta(k_0 + k)}\right\| + \frac{1}{2\eta(k_0 + k)}\left\|I_{N-1} - \frac{Z^\top LZ}{\zeta}\right\| \\ &\leq 1 - \frac{1}{2\eta(k_0 + k)} + \frac{1 - \eta}{2\eta(k_0 + k)} \\ &= 1 - \frac{1}{2(k_0 + k)}.\end{aligned} \tag{14}$$

Using this relation, we derive

$$\begin{aligned}\|\Phi(k-1,n+1)\| &= \prod_{m=n+1}^{k-1}\left\|I_{N-1}-\frac{Z^\top LZ}{2\eta\zeta(k_0+m)}\right\| \\ &\le \prod_{m=n+1}^{k-1}\left(1-\frac{1}{2(k_0+m)}\right) \\ &\le \exp\left(-\frac{1}{2}\sum_{m=n+1}^{k-1}\frac{1}{k_0+m}\right) \\ &\le \exp\left(-\frac{1}{2}\sum_{m=k_0+n+1}^{k_0+k-1}\frac{1}{m}\right) \\ &\le \sqrt{\frac{k_0+n+1}{k_0+k}}. \end{aligned} \tag{15}$$

We also derive

$$\begin{aligned}&\mathsf{Tr}\left(\mathsf{Cov}[\delta_a(k)]\right) \\ &= \mathsf{Tr}\left(\mathsf{E}\left[\sum_{n=1}^{k-1} r(n)^2\Phi(k-1,n+1)\tilde{w}_a(n)\right.\right. \\ &\qquad\qquad \left.\left.\tilde{w}_a(n)^\top\Phi(k-1,n+1)^\top\right]\right) \\ &\le \mathsf{Tr}(N\sigma^2 I_{N-1})\sum_{n=1}^{k-1} r(n)^2\,\|\Phi(k-1,n+1)\|^2 \\ &\le \frac{N(N-1)\sigma^2}{4\eta^2\zeta^2}\sum_{n=1}^{k-1}\frac{1}{(k_0+n)^2}\left(\frac{k_0+n+1}{k_0+k}\right) \\ &\le \frac{N(N-1)\sigma^2}{4\eta^2\zeta^2}\sum_{n=1}^{k-1}\frac{4}{(k_0+n+1)^2}\left(\frac{k_0+n+1}{k_0+k}\right) \\ &\le \frac{N(N-1)\sigma^2}{\eta^2\zeta^2(k_0+k)}\sum_{n=k_0+1}^{k_0+k-1}\frac{1}{n+1} \\ &\le \frac{N(N-1)\sigma^2}{\eta^2\zeta^2(k_0+k)}\ln\frac{k_0+k}{k_0+1} \\ &\le \frac{N(N-1)\sigma^2}{\eta^2\zeta^2\sqrt{(k_0+k)(k_0+1)}} \\ &\le \frac{N(N-1)\sigma^2}{\eta^2\zeta^2\sqrt{(k_0+\tau_2)(k_0+1)}} = \beta^2\gamma. \end{aligned} \tag{16}$$

By (15) of the case $n=0$,

$$\|\Phi(k-1,1)\| \le \sqrt{\frac{k_0+1}{k_0+k}} \tag{17}$$

holds. Then, we have

$$\begin{aligned}\|\mathsf{E}[\delta_a(k)]\| &= \|\Phi(k-1,1)\delta_a(1)\| \\ &\le \|\Phi(k-1,1)\|\,\|\delta_a(1)\| \\ &\le \sqrt{\frac{k_0+1}{k_0+k}}\|\delta_a(1)\| \\ &\le \sqrt{\frac{k_0+1}{k_0+\tau_1}}\|\delta_a(1)\| = \alpha\|\delta_a(k)\|. \end{aligned} \tag{18}$$

To substitute (16) and (18) into (13) and recall that $\|\delta_a(k)\| = \|\delta(k)\|$, Theorem 1 is derived.

V. CONCLUSION

This paper analyzed the convergence of the consensus problem in multi-agent systems with directed noisy communication. The relation between the closeness of the agreement and the number of iterations of the consensus algorithm is clarified explicitly. Unlike the case of undirected graph topology, it is not directly characterized by the eigenvalue of the graph Laplacian. Nonetheless, it is enough to estimate the number of iterations to achieve the desired closeness of the agreement.

REFERENCES

[1] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.

[2] W. Ren, R. Beard, and E. Atkins, "A survey of consensus problems in multi-agent coordination," in *Proceedings of the 2005 American Control Conference*, pp. 1859–1864, 2005.

[3] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*. Princeton University Press, 2009.

[4] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[5] M. Huang and J. Manton, "Stochastic consensus seeking with noisy and directed inter-agent communication: Fixed and randomly varying topologies," *IEEE Transactions on Automatic Control*, vol. 55, no. 1, pp. 235–241, 2010.

[6] T. Li and J.-F. Zhang, "Consensus conditions of multi-agent systems with time-varying topologies and stochastic communication noises," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2043–2057, 2010.

[7] L. Xiao, S. Boyd, and S.-J. Kim, "Distributed average consensus with least-mean-square deviation," *Journal of Parallel and Distributed Computing*, vol. 67, no. 1, pp. 33–46, 2007.

[8] D. Zelazo and M. Mesbahi, "Edge agreement: Graph-theoretic performance bounds and passivity analysis," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 544–555, 2011.

[9] R. Morita, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki, "Convergence analysis of multi-agent consensus with noisy communication," in *the 45th ISCIE International Symposium on Stochastic Systems Theory and Its Applications*, 2013

[10] H.-F. Chen, *Stochastic Approximation and Its Applications*. Kluwer Academic Publishers, 2002.

[11] J. C. Spall, *Introduction to Stochastic Search and Optimization*. John Wiley & Sons, 2003.

[12] T. Wada, T. Itani, and Y. Fujisaki, "A stopping rule for linear stochastic approximation," in *Proceedings of the 49th IEEE Conference on Decision and Control*, pp. 4171–4176 2010.

[13] T. Wada and Y. Fujisaki, "A stopping rule for simultaneous perturbation stochastic approximation," in *Proceedings of the 12th European Control Conference*, pp. 644–649, 2013.