

On formation control of agents subject to ideal Coulomb friction*

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Abstract—This paper deals with the formation control of a group of point masses subject to ideal Coulomb friction. We propose a discontinuous controller which provides a solution to the exact formation problem in the presence of ideal Coulomb friction.

I. INTRODUCTION

Formation keeping control is a cooperative control problem which aims at achieving a desired collective behavior, mainly forming a desired geometrical shape, for the agents of a multi-agent system. Formation control of a group of point masses subject to non-ideal Coulomb friction of the form $f = \frac{\beta}{\sqrt{\alpha + \dot{q}^2}}$, with α, β some constants and \dot{q} the agents velocity, was discussed in [5].

In this paper we show that the continuous controller presented in [5] is not able to achieve exact formations when the agents are subject to ideal Coulomb friction of the form $f = d \operatorname{sign} \dot{q}$. Following [4], we propose a discontinuous controller to reach the desired formation. In [4] exact formation control of a group of double integrators with very coarse exchanged information has been presented. Also, saturated and coarse sensing scenarios for the agents' velocities have been treated. In the current paper we show that the discontinuous controller from [4] achieves exact formation control for agents subject to ideal Coulomb friction, under some conditions on the gain of the controller.

Furthermore we give a physical interpretation of the problem and the controller within the port-Hamiltonian framework. Moreover, we compare the results of the discontinuous controller with the continuous controller.

The outline of the paper is as follows. First we briefly recall some basic preliminaries for port-Hamiltonian systems and graph theory, which are used in the remainder of the paper. Section II provides a motivating example for one agent, showing why discontinuous control can overcome ideal Coulomb friction, while continuous control can not. We continue with the multi-agent setup (Section III) and

provide simulation results illustrating the effectiveness of the approach (Section IV). The paper is concluded in Section V.

A. Preliminaries

The port-Hamiltonian framework is an energy-based modeling framework which describes a large class of (nonlinear) multi-domain systems [3]. Consider state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $y \in \mathbb{R}^m$, skew-symmetric structure matrix $J(x) \in \mathbb{R}^{n \times n}$, positive semi-definite dissipation matrix $R(x) \in \mathbb{R}^{n \times n}$, input matrix $g(x) \in \mathbb{R}^{n \times m}$, and let $H(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the Hamiltonian of the system. The port-Hamiltonian dynamics are given by

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u + c(x)u^r \\ y &= g^T(x) \frac{\partial H}{\partial x}(x) \\ y^r &= c^T(x) \frac{\partial H}{\partial x}(x), \end{aligned} \quad (1)$$

where (u^r, y^r) denotes a resistive port such that $y^{rT} u^r \leq 0$.

The exchange of information between agents is modeled by an undirected connected graph $G(V, E)$, where node-set V corresponds to n agents and the edge-set $E \subset V \times V$ corresponds to m virtual springs. Assign an orientation to each edge by labeling one end with a positive sign and the other end with a negative sign. The incidence matrix B associated to $G(V, E)$ describes which nodes are coupled by an edge, and is defined as

$$b_{i\ell} = \begin{cases} +1 & \text{if node } i \text{ is at the positive end of the edge } \ell \\ -1 & \text{if node } i \text{ is at the negative end of the edge } \ell \\ 0 & \text{otherwise.} \end{cases}$$

II. CONTROL OF A SINGLE POINT MASS SUBJECT TO IDEAL COULOMB FRICTION

Consider a single point mass m subject to ideal Coulomb friction. Let $q \in \mathbb{R}$ denote the position and $p = m\dot{q} \in \mathbb{R}$ the corresponding momentum. The dynamics of such a system are given by

$$\begin{aligned} \dot{q} &= \frac{p}{m} \\ \dot{p} &= -d \operatorname{sign} \frac{p}{m} + u, \end{aligned} \quad (2)$$

where $d \operatorname{sign} \frac{p}{m}$ refers to ideal Coulomb friction with friction coefficient d . Note that (2) can be easily presented in the port-Hamiltonian form (1) by defining

$$J(q, p) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g(q, p) = c(q, p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

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$R(q, p) = 0$, and $H(p) = \frac{1}{2m}p^2$. The resistive port is defined as $u^r = -d \operatorname{sign} y^r$, such that $y^r u^r \leq 0$.

The control target is to stabilize the system's equilibrium, $(q, p) = 0$, by means of a virtual spring. We show that the control objective can not be achieved using a continuous spring, while it can be achieved using a discontinuous spring. We consider two controllers: a continuous spring with corresponding spring force $u_1 = -kq$, and a discontinuous spring with corresponding spring force $u_2 = -k \operatorname{sign} q$. Here $\operatorname{sign} q$ is defined as

$$\operatorname{sign} q = \begin{cases} +1 & \text{if } q \geq 0 \\ -1 & \text{if } q < 0 \end{cases}.$$

Since the right-hand side of (2) is discontinuous, due to the discontinuity of sign at zero, we first adopt an appropriate notion of solution. In this paper, we adopt a Krasovskii notion of solution [1].

We define the set-valued map corresponding to (2) with $u = u_1$ as

$$F_1(q, p) = \left(\begin{array}{c} \frac{p}{m} \\ -kq - d \mathcal{K} \operatorname{sign} \frac{p}{m} \end{array} \right) \quad (3)$$

where

$$\mathcal{K} \operatorname{sign} \frac{p}{m} = \begin{cases} \{\operatorname{sign} \frac{p}{m}\} & \text{if } p \neq 0 \\ [-1, 1] & \text{if } p = 0 \end{cases}. \quad (4)$$

Similarly, for $u = u_2$, we define the set-valued map, $F_2(q, p)$ as

$$F_2(q, p) = \left(\begin{array}{c} \frac{p}{m} \\ -k \mathcal{K} \operatorname{sign} q - d \mathcal{K} \operatorname{sign} \frac{p}{m} \end{array} \right), \quad (5)$$

with $\mathcal{K} \operatorname{sign} \frac{p}{m}$ defined in (4), and $\mathcal{K} \operatorname{sign} q$ is defined as

$$\mathcal{K} \operatorname{sign} q = \begin{cases} \{\operatorname{sign} q\} & \text{if } q \neq 0 \\ [-1, 1] & \text{if } q = 0. \end{cases} \quad (6)$$

Local existence of Krasovskii solutions to the differential inclusion above is guaranteed [1]. We are now ready to show that for $u = u_1$ system (2) does not converge to the origin $(q, p) = 0$, while for $u = u_2$ it does.

Proposition 1: Any Krasovskii solution to (2) exists for all $t \geq 0$. With $u = u_1$ all Krasovskii solutions converge to the largest weakly invariant set where $p = 0$ and $q \in [-\frac{d}{k}, \frac{d}{k}]$, while for $u = u_2$ all the solutions converge to the origin.

Proof: Consider $u = u_1$ and the closed-loop Hamiltonian

$$H_1(q, p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$$

as a candidate Lyapunov function, where the second term on the right-hand side corresponds to the potential energy stored in the continuous virtual spring. We calculate the set-valued derivative $\overline{H}_1(q, p)$ along the solutions of the system. The set-valued derivative $\overline{H}_1(q, p)$ is defined as [1]

$$\begin{aligned} \overline{H}_1(q, p) &= \{a \in \mathbb{R} : \exists w \in F_1(q, p) \text{ s.t.} \\ &a = \langle w, v \rangle, \text{ for all } v \in \nabla H_1(q, p)\}. \end{aligned}$$

By definition of $F_1(q, p)$ in (3), for any $w \in F_1(q, p)$ there exists $w^p \in \mathcal{K} \operatorname{sign} \frac{p}{m}$. Therefore, $\overline{H}_1(q, p) = \{-\frac{d}{m} p w^p\}$. Since $w^p \in \mathcal{K} \operatorname{sign} \frac{p}{m}$, we conclude $\overline{H}_1(q, p) \subset (-\infty, 0]$. Applying LaSalle's invariance principle for non-smooth systems [2], the solutions of the system converge to the largest weakly invariant set where $p = 0$, hence any point $(q, 0)$ on this invariant set should satisfy

$$\left\{ (q, 0) \mid 0 = \begin{pmatrix} 0 \\ -dw^p - kq \end{pmatrix} \right\}. \quad (7)$$

Since $w^p \in \mathcal{K} \operatorname{sign} \frac{p}{m}$, the above expression implies

$$\left\{ (q, 0) \mid \mathbf{0} \in \begin{pmatrix} 0 \\ -d[-1, 1] - kq \end{pmatrix} \right\} \quad (8)$$

which implies that $q \in [-\frac{d}{k}, \frac{d}{k}]$. Therefore, the control objective is not achieved. Now, consider the discontinuous spring with the corresponding control law $u = u_2$. The closed-loop Hamiltonian is a locally Lipschitz function

$$H_2(q, p) = \frac{1}{2m}p^2 + k|q|,$$

where $|\cdot|$ denotes the one-norm and $k|q|$ corresponds to the potential energy stored in the discontinuous spring. From (5), for any $w \in F_2(q, p)$ there exists a $w^p \in \mathcal{K} \operatorname{sign} \frac{p}{m}$ and a $w^q \in \mathcal{K} \operatorname{sign} q$ such that

$$w = \begin{pmatrix} \frac{p}{m} \\ -k w^q - d w^p \end{pmatrix}.$$

Using the tools of non-smooth analysis (Clarke generalized gradient [1]), we calculate the gradient

$$\partial H_2(q, p) = \{v : v = \begin{pmatrix} k v^q \\ \frac{p}{m} \end{pmatrix} \text{ s.t. } v^q \in \mathcal{K} \operatorname{sign} q\},$$

and the set-valued derivative $\overline{H}_2(q, p)$, as

$$\begin{aligned} \overline{H}_2(q, p) &= \{a \in \mathbb{R} : \exists w \in F_2(q, p) \text{ s.t.} \\ &a = \langle w, v \rangle, \text{ for all } v \in \partial H_2(q, p)\}. \end{aligned}$$

Choose $v^q = w^q$. Hence, $\overline{H}_2(q, p) = \{-\frac{d}{m} p w^p\}$ (for the sake of brevity, details of the calculations are omitted). Similar to the previous case we conclude that $\overline{H}_2(q, p) \subset (-\infty, 0]$. By LaSalle's invariance principle for non-smooth systems, the solutions of the system converge to the largest weakly invariant set where $p = 0$, hence any point $(q, 0)$ on this invariant set should satisfy

$$\left\{ (q, 0) \mid 0 = \begin{pmatrix} 0 \\ -dw^p - kw^q \end{pmatrix} \right\}.$$

Let $k > d$. We claim that on the invariant set q necessarily must be zero. By contradiction, assume that q is not zero. Hence, $q > 0$ or $q < 0$ and the largest weakly invariant set is

$$\left\{ (q, 0) \mid \mathbf{0} \in \begin{pmatrix} 0 \\ -d[-1, 1] - kw^q \end{pmatrix} \right\} \quad (9)$$

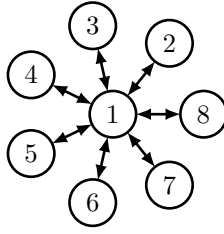


Fig. 1. Star graph for $n = 8$ agents.

where w^q is either 1 or -1 . If $w^q = 1$ the second element of the above set is $[-(d+k), d-k]$. Since $k > d$, both of the upper and lower bound of this closed interval are negative. Hence, zero can not belong to this interval. A similar argument can be done for $w^q = -1$, where both of the bounds of the interval are now positive, which contradicts the requirement for the points on the invariant set. Therefore, q must be zero. ■

Remark 1 (Physical interpretation of the result): With the continuous spring the position q and consequently the spring force kq converge to zero while the agent position is converging to the target position $q = 0$. Therefore, when $-\frac{d}{k} < q < \frac{d}{k}$ the spring force can no longer overcome the friction force and the agent will stop before reaching the target position $q = 0$. On the other hand, the discontinuous spring always provides a spring force of either $+k$ or $-k$, no matter how close the agent is to the target position. Therefore if $k > d$, the target can be achieved.

III. FORMATION CONTROL OF A NETWORK OF n AGENTS SUBJECT TO IDEAL COULOMB FRICTION

In this section we benefit from the results of Section II to analyze formation control of a network of n point masses subject to ideal Coulomb friction. Let $q \in \mathbb{R}^n$, $p \in \mathbb{R}^n$ denote the positions and momenta of the agents. Furthermore let $M = \text{diag}(m_1, \dots, m_n) \in \mathbb{R}^{n \times n}$, $D = d I_n \in \mathbb{R}^{n \times n}$.

The exchange of information between agents is modeled by a star graph (see Fig. 1). The inter-agent (relative) position vector is defined as $z = B^T q$, where B denotes the incidence matrix of the graph. The control target is to achieve a consensus for the inter-agent positions. The closed-loop dynamics of the network are given by

$$\begin{aligned} \dot{z} &= B^T M^{-1} p \\ \dot{p} &= -D \text{sign} M^{-1} p + u, \end{aligned} \quad (10)$$

which can be written in the port-Hamiltonian form (1) by defining

$$J(z, p) = \begin{pmatrix} 0 & B^T \\ -B & 0 \end{pmatrix}, \quad g(z, p) = c(z, p) = \begin{pmatrix} 0 \\ I_n \end{pmatrix},$$

$R(z, p) = 0$, and $H(p) = \frac{1}{2} p^T M^{-1} p$. The resistive port is defined as $u^r = -D \text{sign} y^r$, such that $y^r T u^r \leq 0$. As in Section II we consider a continuous controller $u_1 = -BKz$ and a discontinuous controller $u_2 = -BK \text{sign} z$, where z denotes the relative position and $K = k I_m \in \mathbb{R}^{m \times m}$.

Now, we prove that system (10) does not converge to the origin with $u = u_1$, but it does converge to the origin $(z, p) = \mathbf{0}$ with $u = u_2$. Similar to the previous section, we adopt a Krasovskii notion of solution since the right-hand side of (10) is discontinuous.

Proposition 2: Any Krasovskii solution to (10) exists for all $t \geq 0$. With $u = u_1$ all Krasovskii solutions converge to the largest weakly invariant set where $p = \mathbf{0}$ and $z_\ell \in [-\frac{d}{k}, \frac{d}{k}]$ for $\ell = 1, \dots, m$ and with $u = u_2$ the solutions do converge to the origin $(z, p) = \mathbf{0}$.

Proof: Consider the control law corresponding to continuous springs. Let $u = u_1$, hence, the closed-loop dynamics of the network are given by

$$\begin{aligned} \dot{z} &= B^T M^{-1} p \\ \dot{p} &= -D \text{sign} M^{-1} p - BKz. \end{aligned} \quad (11)$$

Take the Hamiltonian $H_1(z, p) = \frac{1}{2} p^T M^{-1} p + \frac{1}{2} z^T K z$ as the Lyapunov candidate, where $\frac{1}{2} z^T K z$ corresponds to the total potential energy stored in the m continuous virtual springs. Calculating the set-valued derivative of $H_1(z, p)$ (as in the previous section), we obtain $\overline{H}_1(z, p) = \{-p^T M^{-1} D w^p\} \subset (-\infty, 0]$, where $w^p \in \mathcal{K} \text{sign} p$ is an n -dimensional vector. Characterizing the largest weakly invariant set at which $p = \mathbf{0}$, we have

$$\left\{ (z, \mathbf{0}) \mid \mathbf{0} \in \begin{pmatrix} \mathbf{0} \\ -D w^p - KBz \end{pmatrix} \right\} \quad (12)$$

Label the agent in the center of the star graph, as agent 1. The dynamics of each of the agents can be written as

$$\begin{aligned} \dot{p}_1 &= -d \text{sign} \frac{p_1}{m_1} - k \sum_{\ell} b_{1\ell} z_\ell, \quad i = 1 \\ \dot{p}_i &= -d \text{sign} \frac{p_i}{m_i} - k b_{i\ell} z_\ell, \quad i \neq 1. \end{aligned} \quad (13)$$

From (12) each agent on the invariant set should satisfy $-d w_i^p - k b_{i\ell} z_\ell = 0$ for $i \neq 1$. Also, from (13) it can be seen that the dynamics of \dot{p}_i for $i \neq 1$, obeys the same dynamics as \dot{p} in (2) with $u = kq$. Therefore, using a similar argument as in the previous section, we conclude $z_\ell \in [-\frac{d}{k}, \frac{d}{k}]$ for $\ell = 1, \dots, m$. Hence, the objective is not achieved.

Now, consider the discontinuous controller and let $u = u_2$. The closed-loop dynamics are given by

$$\begin{aligned} \dot{z} &= B^T M^{-1} p \\ \dot{p} &= -D \text{sign} M^{-1} p - BK \text{sign} z. \end{aligned} \quad (14)$$

Take $H_2(z, p) = \frac{1}{2} p^T M^{-1} p + k \sum_{\ell} |z_\ell|$, where $k \sum_{\ell} |z_\ell|$ corresponds to the total potential energy stored in the m discontinuous virtual springs. Applying the same nonsmooth analysis tools as before, it follows that system (11) converges asymptotically to the largest weakly invariant set where $p = \mathbf{0}$. Characterizing the largest weakly invariant set at which $p = \mathbf{0}$, we have

$$\left\{ (z, \mathbf{0}) \mid \mathbf{0} \in \begin{pmatrix} \mathbf{0} \\ -d w^p - kBw^z \end{pmatrix} \right\} \quad (15)$$

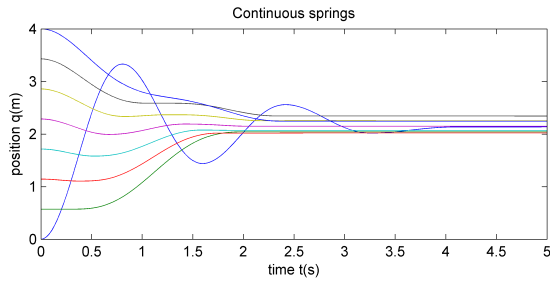


Fig. 2. Time evolution of the agents' positions using continuous springs.

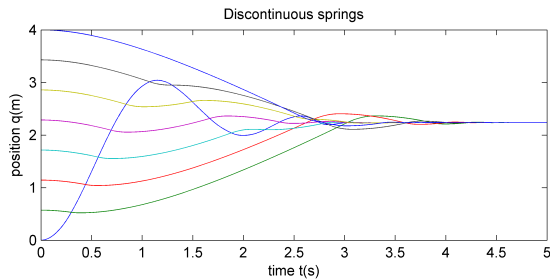


Fig. 3. Time evolution of the agents' positions using discontinuous springs.

where $w^p \in \mathcal{K} \operatorname{sign} M^{-1}p$, and $w^z \in \mathcal{K} \operatorname{sign} z$ are n -dimensional vectors. As explained before, the dynamics of \dot{p}_i , $i \neq 1$ obey (2) for $u = k \operatorname{sign} q$. Hence using a similar argument as in Section II, we conclude that z_ℓ is necessarily equal to zero for $\ell = 1, \dots, m$. ■

IV. SIMULATION RESULTS

We simulate a network of $n = 8$ unit point masses interconnected by a star graph with agent 1 at the center of the graph (see Fig. 1). The Coulomb friction coefficient is set at $d = 1$ and the virtual spring constant is set at $k = 2$.

The outcome for the continuous and discontinuous springs are presented in Fig. 2 and Fig. 3 respectively.

As expected from the previous section, the continuous springs cannot achieve the exact formation control (Fig. 2), while the discontinuous springs can (Fig. 3).

V. CONCLUSION

In this paper we presented a comparison between a continuous and a discontinuous spring in order to control a point mass in \mathbb{R} subject to ideal Coulomb friction. We showed that position control can be achieved with a discontinuous spring, but not with a continuous one. Moreover, we applied the discontinuous controller in order to achieve formation control of a network of n point masses subject to ideal Coulomb friction which are interconnected using a star graph. Future avenues are to consider more general graph topologies, agents evolving in the plane (\mathbb{R}^2), and heterogeneous network dynamics.

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