

# An internal model approach to (optimal) frequency regulation in power grids\*

M. Bürger<sup>1</sup>, C. De Persis<sup>2</sup> and S. Trip<sup>2</sup>

**Abstract**—This paper studies the problem of frequency regulation in power grids under unknown and possible time-varying load changes, while minimizing the generation costs in some particular cases. We formulate this problem as an optimal output agreement problem for distribution networks. This problem has been recently addressed by some of the authors using incremental passivity and dynamic internal-model-based controllers. We believe that this framework is general enough to allow for more complex control scenarios in future extensions.

## I. INTRODUCTION

The power grid can be regarded as a large interconnected network of different subsystems, called control area's. In order to guarantee reliable operation the frequency is tightly regulated around its nominal value, e.g. 50Hz. Automatic regulation of the frequency in power grid is traditionally achieved by primary proportional control (droop-control) and a secondary PI-control. In this secondary control, commonly known as automatic generation control (AGC), each control area determines its "Area Control Error" (ACE) and changes its production accordingly to compensate for local load changes to regulate the frequency back to its nominal value and to maintain the scheduled power flows between different area's.

By requiring each control area to compensate for their local load changes the possibility to achieve economic efficiency is lost. Indeed the scheduled production in the different control area's is currently determined by economic criteria relatively long in advance. To be economically efficient an accurate prediction of load changes is necessary. Large scale introduction of volatile renewable energy sources and the use of electrical vehicles will however make accurate prediction difficult as the net load (demand minus renewable generation) will change on faster time scales and by larger amounts.

An up-to-date review of current research on AGC can be found in [1]. We notice that the economic efficiency of AGC is largely neglected. Two notable exceptions are [2] and [3],

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<sup>1</sup>M. Bürger is with the Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany. [mathias.buerger@ist.uni-stuttgart.de](mailto:mathias.buerger@ist.uni-stuttgart.de)

<sup>2</sup>C. De Persis and S. Trip are with the Institute of Technology and Management, Faculty of Mathematics and Natural Sciences, University of Groningen, Nijenborgh 4, 9747 AG Groningen, the Netherlands. [c.de.persis@rug.nl](mailto:c.de.persis@rug.nl), [s.trip@rug.nl](mailto:s.trip@rug.nl)

where a distributed generation controller that regulates the frequency back to its nominal value and at the same time minimizes the generation costs is proposed.

In our work we address a similar control problem but with an approach that differs substantially from the two aforementioned works. We move along the lines of [4], [5], where an approach to deal with nonlinear output agreement and optimal flow problems for dynamical networks has been proposed. In those papers internal-model-based dynamic controllers have been designed to solve output agreement problems for networks of incrementally passive systems ([6], [7]) in the presence of time-varying perturbations. Controllers that steer the solutions to optimal steady state solutions were also discussed and connections with optimal routing problems for inventory control problems ([8]) and droop controllers in micro-grids ([9]) were established.

In the present paper we continue the investigation of [4], [5] focusing on problems of frequency regulation with economic considerations. In contrast with previous work, we consider perturbation (unknown power demand) that are possibly time-varying and adopt arguments that do not necessarily rely on the linearity of the systems under consideration.

Besides the two features just recalled, our approach based on passivity properties of the system has the potential to deal with more complex models of dynamical networks ([10]) and fairly rich classes of external perturbations ([11]), thus paving the way towards optimal frequency regulators in the presence of a large variety of consumption behaviors. Furthermore, other extensions of [5] considered the presence of non-quadratic cost functions and flow capacity constraints ([12]) that could turn out to be useful also for the problem considered here.

The paper is organized as follows. In Section II, we introduce the dynamic model of the power grid and formalize the problem treated in this paper. In Section III, we analyze the dynamic model with constant generation and show that it leads to a nonzero frequency deviation. In Section IV, we characterize the optimum generation to minimize the generation costs. In section V, we propose a distributed controller which ensures frequency regulation and at the same time minimizes the generation costs under the assumption of constant demands. In Section VI, the restriction of constant demand is relaxed and we extend results to the case of a certain class of time-varying demands. In Section VII, we test our controllers for an academic case study using simulations. In Section VIII, conclusions are given and an outline for future research is provided. Most of the proofs are omitted and can be found in [13].

## II. SYSTEM MODEL AND PROBLEM FORMULATION

### A. System model

Consider a power grid consisting of  $n$  buses. The network is represented by a connected and undirected graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where the nodes,  $\mathcal{V} = \{1, \dots, n\}$ , represent the buses and the edges,  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V} = \{1, \dots, m\}$ , the transmission lines connecting the buses. The network structure can be represented by its corresponding incidence matrix  $D \in \mathbb{R}^{n \times m}$ . The ends of edge  $k$  are arbitrary labeled with a '+' and a '-'. Then

$$d_{ik} = \begin{cases} +1 & \text{if } i \text{ is the positive end of } k \\ -1 & \text{if } i \text{ is the negative end of } k \\ 0 & \text{otherwise.} \end{cases}$$

Each bus represents a control area and is assumed to have controllable power generation and an unknown and uncontrollable load. The dynamics at each bus are given by [14]

$$\begin{aligned} \dot{\delta}_i &= \omega_i^b \\ M_i \dot{\omega}_i^b &= u_i - \sum_{j \in \mathcal{N}_i} V_i V_j B_{ij} \sin(\delta_i - \delta_j) \\ &\quad - A_i (\omega_i^b - \omega^n) - P_i^l \end{aligned} \quad (1)$$

and are commonly known as the swing equations. We make use of the following set of symbols:

$\delta_i$	Voltage angle at bus $i$ ,
$\omega_i^b$	Frequency at bus $i$ ,
$\omega^n$	Nominal frequency,
$V_i$	Voltage at bus $i$ ,
$M_i$	Moment of inertia at bus $i$ ,
$A_i$	Damping constant at bus $i$ ,
$\mathcal{N}_i$	Set of buses connected to bus $i$ ,
$B_{ij}$	Susceptance of the line between buses $i$ and $j$ ,
$P_i^l$	Power demand at bus $i$ ,
$u_i$	Controllable power generation at bus $i$ .

At each bus the power production can be controlled. As different generator units have different cost characteristics we associate to each bus a generation cost function of the form  $C_i(u_i) = q_i u_i^2$ , where  $C_i(u_i)$  is strictly convex, i.e.  $q_i > 0$ .

**Assumption 1** By using system model (1) following assumptions are made, which are standard in a broad range of literature on power grid dynamics.

- Lines are lossless, i.e. the conductance is zero. This assumption is generally valid for the case of high voltage lines connecting different control areas.
- Nodal voltages,  $V_i$  are constant.
- Reactive power flows are ignored.
- A balanced load condition is assumed, such that the three phase network can be analyzed by a single phase.

It is convenient to write system (1) compactly for all buses  $i \in \mathcal{V}$  as

$$\begin{aligned} \dot{\eta} &= D^T \omega \\ M \dot{\omega} &= u - D \Gamma \sin(\eta) - A \omega - P^l, \end{aligned} \quad (2)$$

where  $\omega$  is the frequency deviation  $\omega^b - \mathbf{1}_n \omega^n$ ,  $D$  is the incidence matrix corresponding to the network,  $\Gamma = \text{diag}\{\gamma_1, \dots, \gamma_m\}$  with  $\gamma_k = V_i V_j B_{ij} = V_j V_i B_{ji}$  where  $k$  denotes the line  $\{i, j\}$  and  $\eta = D^T \delta$ . Moreover,  $\sin(\eta) = (\sin \eta_1 \dots \sin \eta_m)^T$ . The total generation costs in the network are given by  $\sum_{i \in \mathcal{V}} q_i u_i^2 = u^T Q u$ , where  $Q = \text{diag}\{q_1, \dots, q_n\}$ .

## III. STABILITY OF THE SWING EQUATIONS

First we investigate the response of system (2) with a constant generation  $\bar{u}$  and constant load  $P^l$ . This corresponds to the case where the generated power is predetermined and is not actively controlled. The problem at hand is to study the attractivity of the steady state solution to (2). This study has been already pursued in several other papers (see e.g. [9], [15]). The novelty in this section is that we investigate the problem within the framework of [4], [5], and we believe this approach can shed new insights into the problem.

In the framework of [4], [5], system (2) is first interpreted as two subsystems interconnected via constraints that reflect the topology of the network. As a matter of fact, observe that system (2) can be viewed as the feedback interconnection of the system

$$\begin{aligned} M \dot{\omega} &= \bar{u} + \mu - A \omega - P^l \\ y &= \omega \end{aligned} \quad (3)$$

with the system

$$\begin{aligned} \dot{\eta} &= v \\ \lambda &= \psi(\eta), \end{aligned} \quad (4)$$

where  $\psi(\eta) = \Gamma \sin(\eta)$ . These systems are interconnected via the relations

$$\begin{aligned} v &= D^T y \\ \mu &= -D \lambda. \end{aligned} \quad (5)$$

Before analyzing the system, it is convenient to recall its equilibria and we do this in the next subsection.

### A. Equilibria

As a first step we characterize the steady state solution  $(\bar{\eta}, \bar{\omega})$  of (2) corresponding to the case in which  $\bar{\omega}$  is a constant belonging to the space  $\mathcal{N}(D^T)$ , i.e. it is a constant vector with all elements being equal. The steady state solution necessarily satisfies

$$\begin{aligned} \mathbf{0} &= D^T \bar{\omega} \\ \mathbf{0} &= \bar{u} - D \Gamma \sin(\bar{\eta}) - A \bar{\omega} - P^l. \end{aligned} \quad (6)$$

Notice that  $\bar{\eta}$  is the vector of relative voltage angles that guarantee the power exchange among the buses. The equilibria can be characterized as follows:

**Lemma 1** Let

$$\mathcal{D} = \{v \in \mathbb{R}^n : v = D \Gamma \sin(\eta), \eta \in \mathcal{R}(D^T)\}.$$

If the input  $\bar{u} - P^l$  is such that

$$\left( I - \frac{A \mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T A \mathbf{1}_n} \right) (\bar{u} - P^l) \in \mathcal{D}, \quad (7)$$

then the equilibria  $(\bar{\eta}, \bar{\omega})$  for which equation (6) is fulfilled are given by

$$\bar{\omega} = \mathbf{1}_n \omega_*, \quad D\Gamma \sin(\bar{\eta}) = \left( I - \frac{A\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T A \mathbf{1}_n} \right) (\bar{u} - P^l) \quad (8)$$

where

$$\omega_* = \frac{\mathbf{1}_n^T (\bar{u} - P^l)}{\mathbf{1}_n^T A \mathbf{1}_n} = \frac{\sum_{i \in \mathcal{V}} (\bar{u}_i - P_i^l)}{\sum_{i \in \mathcal{V}} A_i} \quad (9)$$

**Remark 1** The characterization of the equilibria (8) has already been given in a number of forms (e.g. [9], [15]). The feasibility condition (7) is a strong condition which is known to be hard to satisfy. See [15], Assumption 5.3, for an analogous condition.

### B. Stability

Having characterized the steady state solution of the system (2) and having assumed that such a steady state solution exists, we are ready to state a result concerning its attractivity. Although the result is not new, its proof (which is reported in [13]) helps us in setting the ground for later analysis and it is therefore reported for the sake of completeness. Moreover, the proof explores the role of internal model arguments and this role is further extended in other results in this paper.

The proof can be split in a number of basic steps. First, one can show that systems (3), (4) are incrementally passive with respect to the equilibrium solution, namely:

**Proposition 1** System (3) is an output strictly incrementally passive system with respect to a constant equilibrium pair  $(\bar{\omega}, \bar{\mu})$ , namely there exists a regular storage function  $V(\omega, \bar{\omega})$  which satisfies the incremental dissipation inequality  $\dot{V}(\omega, \bar{\omega}) = -\rho(y - \bar{y}) + (y - \bar{y})^T(\mu - \bar{\mu})$ , where  $\dot{V}$  represents the directional derivative of  $V$  along the solutions to (3) and  $\rho$  is a positive definite function.

*Proof:* Consider the regular storage function

$$V(\omega, \bar{\omega}) = \frac{1}{2} (\omega - \bar{\omega})^T M (\omega - \bar{\omega}).$$

We have

$$\begin{aligned} \dot{V} &= (\omega - \bar{\omega})^T (\bar{u} + \mu - A\omega - P^l) \\ &= (\omega - \bar{\omega})^T (-A(\omega - \bar{\omega}) + (\mu - \bar{\mu})) \\ &= -(y - \bar{y})^T A(y - \bar{y}) + (y - \bar{y})^T (\mu - \bar{\mu}), \end{aligned}$$

which proves the claim.  $\square$

**Proposition 2** System (4) is an incrementally passive system with respect to a constant equilibrium  $\bar{\eta}$ , namely there exists a regular storage function  $W(\eta, \bar{\eta})$  which satisfies the incremental dissipation inequality  $\dot{W}(\eta, \bar{\eta}) = (\lambda - \bar{\lambda})^T (v - \bar{v})$ , where  $\dot{W}$  represents the directional derivative of  $W$  along the solutions to (4).

*Proof:* Consider the storage function

$$W(\eta, \bar{\eta}) = \Psi(\eta) - \Psi(\bar{\eta}) - \nabla \Psi(\bar{\eta})^T (\eta - \bar{\eta})$$

having denoted with  $\Psi(\eta)$  the function such that  $\nabla \Psi(\eta) = \psi(\eta)$ , i.e.  $\Psi(\eta) = -\mathbf{1}^T \Gamma \cos(\eta)$ , and with  $\bar{\eta}$  the signal introduced in Lemma 1 and satisfying  $\bar{\eta} = \mathbf{0}$ ,  $\psi(\bar{\eta}) = \Gamma \sin(\bar{\eta})$ . Bearing in mind [16], [17], [4], [5], it can be shown that  $W(\eta, \bar{\eta}) \geq 0$  for all  $\eta$  in a neighborhood of  $\bar{\eta}$ , with  $W(\eta, \bar{\eta}) = 0$  if and only if  $\eta = \bar{\eta}$ . The dissipation inequality writes as

$$\begin{aligned} \dot{W}(\eta, \bar{\eta}) &= (\nabla \Psi(\eta) - \nabla \Psi(\bar{\eta}))^T \dot{\eta} \\ &= (\lambda - \bar{\lambda})^T (v - \bar{v}) \end{aligned}$$

where the last equality trivially holds since  $\dot{\eta} = \bar{v} = \mathbf{0}$ . This proves the claim.  $\square$

The interconnection of incrementally passive systems via (5) is known to be still incrementally passive and this fact is used to prove convergence to the steady state solution of the system, resulting in Theorem 1. Details are provided in the extended version [13]. We notice that a similar study has been investigated in [5], Section 7, for a network of first-order oscillators connected to frequency droop controllers. Here the study is carried out for the second-order swing equations (2).

**Theorem 1** There exists a neighborhood of initial conditions around an equilibrium  $(\bar{\eta}, \bar{\omega})$ , such that the solutions to (2) starting from this neighborhood converge asymptotically to an equilibrium satisfying (8), (9).

In the previous result we set  $u = \bar{u}$  in the swing equations. If we now let  $u$  be any control input then one can prove that the overall system is incrementally passive from the input  $u - \bar{u}$  to the output  $y - \bar{y}$ . In fact, we have

**Corollary 1** For system (2) there exists a regular storage function  $U(\omega, \bar{\omega}, \eta, \bar{\eta})$  which satisfies the following incremental dissipation inequality

$$\dot{U}(\omega, \bar{\omega}, \eta, \bar{\eta}) = -\rho(y - \bar{y}) + (y - \bar{y})^T (u - \bar{u})$$

where  $\dot{U}$  represents the directional derivative of  $U$  along the solutions to (2) and  $\rho$  is a positive definite function.

This result states that the “feedback” interconnection (5) preserves incremental passivity with  $u$  as an input. We can exploit this feature to further design incrementally passive controllers that generate  $u$  while establishing desired properties for the overall closed-loop system. These additional properties that one can establish are studied in the next sections.

## IV. MINIMIZING GENERATION COSTS

In this section we characterize the optimal generation such that associated costs are minimized. The corresponding network optimization problem tackled is as follows:

$$\begin{aligned} \min_{u, v} C(u) &= \min_{u, v} \sum_{i \in \mathcal{V}} C_i(u_i) \\ \text{s.t.} \quad 0 &= u - D\Gamma v - P^l, \end{aligned}$$

where we have set  $\sin(\eta) = v$ . Notice that the equality constraint in the optimization problem coincides with the dynamics in (2) at steady state with  $\omega = \mathbf{0}$ . Following standard literature on convex optimization, we introduce the Lagrangian function  $L(u, v, \lambda) = C(u) + \lambda^T (u - D\Gamma v - P^l)$ . Assume that  $C(u)$  is strictly convex so that  $L(u, v, \lambda)$  is convex in  $(u, v)$ . Notice that  $L(u, \eta, \lambda)$  is concave in  $\lambda$ . Therefore there exists a saddle point solution to  $\max_{\lambda} \min_{u, v} L(u, v, \lambda)$ . Applying first order optimality conditions, the saddle point  $(\bar{u}, \bar{v}, \bar{\lambda})$  must satisfy

$$\begin{aligned} \nabla C(\bar{u}) + \bar{\lambda} &= \mathbf{0} \\ \Gamma D^T \bar{\lambda} &= \mathbf{0} \\ \bar{u} - D\Gamma \bar{v} - P^l &= \mathbf{0} \end{aligned} \quad (10)$$

We make explicit the solution to the previous set of equations in the case of quadratic cost functions.

**Lemma 2** Let  $C(u) = \frac{1}{2}u^T Qu$ , with  $Q > 0$  and diagonal, and  $\mathcal{D}$  be defined as in Lemma 1. If  $P^l$  is such that  $(Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} - I_n) P^l \in \mathcal{D}$ , then the optimal control is

$$\bar{u} = Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T P^l}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n}, \quad (11)$$

the optimal Lagrange multiplier is  $\bar{\lambda} = \mathbf{1}_n \left( -\frac{\mathbf{1}_n^T P^l}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} \right)$  and an optimal voltage angle displacement is given by the vector  $\bar{\eta} \in \mathcal{R}(D^T)$  that satisfies

$$D\Gamma \sin(\bar{\eta}) = (Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} - I_n) P^l. \quad (12)$$

The proof is standard and is omitted. This result shows that an optimal solution may require a nonzero  $D\Gamma \sin(\bar{\eta})$  at steady state. That implies that at steady state power flows may be exchanged among the buses in the network and that the local demand  $P_i^l$  may not necessarily be all compensated by  $\bar{u}_i$ .

## V. OPTIMAL SECONDARY DYNAMICAL CONTROLLERS

Theorem 1 shows attractivity of the steady state solution under constant power demand  $P^l$ . In this section we consider the problem of designing the production  $u$  in such a way that at steady state the system achieves a zero frequency deviation. Although similar results have appeared in the literature, we contribute a new approach to the problem that rests on incremental passivity and on the use of the internal model principle. We believe this is important because it provides a framework in which similar problems can be addressed (for instance with more accurate models and/or with different classes of external perturbations) and because our framework provides an explicit way to design the control.

We start the analysis first reminding that the Corollary 1 states the incremental passivity property of the system

$$\begin{aligned} \dot{\eta} &= D^T \omega \\ M \dot{\omega} &= u - A\omega - D\Gamma \sin(\eta) - P^l \\ y &= \omega \end{aligned} \quad (13)$$

The incremental passivity property holds with respect to two solutions of (13). As one of the two solutions, we adopt here a steady state solution to the regulator equations. This is the state  $\bar{x} = (\bar{\eta}, \bar{\omega})$ , the feedforward input  $\bar{u}$  and the output  $\bar{y} = \bar{\omega} = 0$  such that

$$\begin{aligned} \dot{\bar{\eta}} &= D^T \bar{\omega} = \mathbf{0} \\ \mathbf{0} &= \bar{u} - D\Gamma \sin(\bar{\eta}) - P^l \\ \bar{y} &= \bar{\omega} = \mathbf{0}. \end{aligned} \quad (14)$$

Among the many possible choices, we focus on the steady state solution that arises from the solution of the optimal control problem in the previous section, namely

$$\bar{u} = Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T P^l}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n}, \quad (15)$$

characterized in (11) above, and  $\bar{\eta}$  such that

$$D\Gamma \sin(\bar{\eta}) = (Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} - I_n) P^l. \quad (16)$$

We are now ready to state the main result of the section, whose proof can be found in the extended version [13]. We propose a dynamic controller that converges asymptotically to the optimal feedforward input that guarantees zero frequency deviation.

**Theorem 2** Given the system (13), with constant power demand  $P^l$ , the controllers at the nodes

$$\begin{aligned} \dot{\theta}_i &= \sum_{j \in \mathcal{N}_i^{comm}} (\theta_j - \theta_i) - q_i^{-1} \omega_i \\ u_i &= q_i^{-1} \theta_i, \quad i = 1, 2, \dots, n, \end{aligned}$$

where  $\mathcal{N}_i^{comm}$  denotes the set of neighbors of node  $i$  in a graph describing the exchange of information among the controllers, guarantee the solutions to the closed-loop system to converge asymptotically to the largest invariant set where  $\omega_i = 0$  for all  $i = 1, 2, \dots, n$ , and  $\theta = \bar{\theta}$ ,  $\bar{\theta}$  being the vector

$$\bar{\theta} = \frac{\mathbf{1}_n \mathbf{1}_n^T P^l}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n}$$

such that  $\bar{u} = Q^{-1} \bar{\theta}$  satisfies

$$\begin{aligned} \dot{\bar{\eta}} &= \mathbf{0} \\ \mathbf{0} &= \bar{u} - D\Gamma \sin(\bar{\eta}) - P^l. \end{aligned}$$

**Remark 2** The use of an auxiliary communication graph to allow the exchange of information among the controllers at the node has been suggested in [9]. While we adopt the same controller structure, here we provide a controller for the second-order swing equations that additionally converge to an optimal steady state and we cast the analysis within the incremental passivity framework of [5].

The interpretation of the theorem is straightforward: it shows that the dynamic controllers based on an internal model design synchronize to an optimal steady state solution of the exosystem that generates the feedforward input able to guarantee a zero frequency deviation.

VI. FREQUENCY REGULATION IN THE PRESENCE OF  
TIME-VARYING POWER DEMAND

Future smart grids should be able to cope with rapid fluctuations of the power demand at the same time scale of the physical infrastructure. This asks for controllers able to deal with time-varying power demand. In the previous section we studied dynamical controllers able to achieve zero frequency deviation with steady state optimal production in the presence of constant power demand. Since the framework of [5] lends itself to deal with time-varying demand, it is natural to wonder whether the approach can be used to design frequency regulators in the presence of time-varying power demand. This is investigated in this section.

Let  $P^l$  depend linearly on  $w$ , namely, let

$$P^l = \Pi w, \quad (17)$$

for some matrix  $\Pi$ , where  $w$  is the state variable of the exosystem

$$\dot{w} = s(w). \quad (18)$$

Here the map  $s$  satisfies the incremental passivity property  $(s(w) - s(w'))^T(w - w') \leq 0$  for all  $w, w'$ . It will be useful to limit ourselves to the case  $s(w) = Sw$ , with  $S$  a skew-symmetric matrix<sup>1</sup>. However, we will continue to refer to  $s(w)$  for the sake of generality, using explicitly  $Sw$  only when needed.

Consider the optimal production

$$\bar{u} = Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T P^l}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n}, \quad (19)$$

characterized in (11) above. In this case, the second equality in (14) writes as in (12)

$$D\Gamma \sin(\bar{\eta}) = (Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} - I_n) P^l. \quad (20)$$

This implies that the quantity on the right-hand side must be constant and that there must exist a vector  $\bar{\eta} \in \mathcal{R}(D^T)$  which satisfies the equality. If in the disturbance term  $\Pi w(t)$  we differentiate between a constant component  $\Pi_1 w_1$  and a time-varying one  $\Pi_2 w_2(t)$ , i.e.  $\Pi w(t) = \Pi_1 w_1 + \Pi_2 w_2(t)$ , and there exists a solution to the identity (20) when  $\Pi w$  is replaced by  $\Pi_1 w_1$ , then such a solution continues to exist provided that the time-varying component of  $\Pi w(t)$  belongs to the null space of  $(Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} - I_n)$ .

The null space above can be easily characterized. In fact, it can be verified that the matrix  $-\mathbf{1}_n^T Q^{-1} \mathbf{1}_n \cdot (Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} - I_n)$  takes the expression

$$L^T = \begin{pmatrix} L_{11}^T & -q_1^{-1} & \dots & -q_1^{-1} \\ -q_2^{-1} & L_{22}^T & \dots & -q_2^{-1} \\ \vdots & \vdots & \vdots & \vdots \\ -q_n^{-1} & -q_n^{-1} & \dots & L_{nn}^T \end{pmatrix}$$

where  $L_{ii}^T = (\sum_{j \in \mathcal{V} \setminus \{i\}} q_j^{-1})$ . Hence,  $L$  is the Laplacian matrix of a weighted complete graph. The rank of the

<sup>1</sup>In this case, the exosystem (18), (17) generates linear combinations of constant and sinusoidal signals.

Laplacian matrix of a connected graph is  $n - 1$ . Thus the rank of the matrix  $L^T$  is  $n - 1$ . Since the rank of a matrix is not altered by the multiplication of a nonzero constant, one infers that the matrix  $(Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} - I_n)$  has rank  $n - 1$  as well. Thus its null space has dimension 1. Now, it is easily checked that the range of  $Q^{-1} \mathbf{1}_n$  is included in the null space of  $(Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} - I_n)$ . Hence, the time varying component  $\Pi_2 w_2(t)$  of the unknown demand must satisfy  $\Pi_2 w_2(t) \in \mathcal{R}(Q^{-1} \mathbf{1}_n)$ . This leads to the following model for the power demand

$$\begin{aligned} \dot{w}_1 &= 0, & \dot{w}_2 &= s_2(w_2) \\ P^l &= \Pi_1 w_1 + Q^{-1} \mathbf{1}_n R w_2, \end{aligned} \quad (21)$$

where  $R$  is some suitable row vector whose properties will be specified later. As a result, if we consider the contribution of the time-varying component of the disturbance to the optimal steady-state controller, it must be true that

$$\bar{u}_2 = Q^{-1} \frac{\mathbf{1}_n \mathbf{1}_n^T \Pi_2 w_2}{\mathbf{1}_n^T Q^{-1} \mathbf{1}_n} = Q^{-1} \mathbf{1}_n R w_2 \quad (22)$$

where we have exploited the identity  $\Pi_2 w_2 = Q^{-1} \mathbf{1}_n R w_2$ . This identity will be used later in the section.

**Example 1** Consider the case of a periodic power demand with frequency  $\mu$  superimposed to a constant power demand. This demand can be modeled as  $P_i^l = \Pi_{1i} w_1 + \Pi_{2i} w_2$  where  $\dot{w} = Sw$ ,

$$S = \begin{pmatrix} 0 & \mathbf{0}_{1 \times 2} \\ \mathbf{0}_{2 \times 1} & S_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu \\ 0 & -\mu & 0 \end{pmatrix},$$

$\Pi_{1i}$  is a real number and  $\Pi_{2i} = q_i^{-1}(1 \ 0)$ . In this case  $R = (1 \ 0)$ . The pair  $(R, S_2)$  is observable.

This characterization points out that, for the existence of a steady state solution with a zero frequency deviation in the presence of time-varying demand, the exchange of power among the different areas must be constant at steady state and this requires that the intensity of the power demand at one aggregate area should be inversely proportional to the power production cost at the same area. We stress that this is not a limitation of the approach in [5], but rather a constraint imposed by the physics of the smart grid and the zero frequency regulation problem.

We are now ready to state the main result of this section, whose proof can be found in the extended version [13]:

**Theorem 3** Suppose that there exists a solution to the regulator equations (14) with  $P^l$  as in (21). Then, given the system (13), with exogenous power demand  $P^l$  generated by (21), with  $s_2(w_2) = S_2 w_2$ ,  $S_2$  skew-symmetric and with purely imaginary eigenvalues<sup>2</sup>, and  $(R, S_2)$  an observable

<sup>2</sup>The zero does not belong to the spectrum of  $S_2$ .

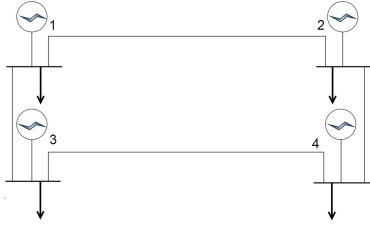


Fig. 1. A 4-area interconnected electricity network.

pair, the controllers at the nodes

$$\begin{aligned} \dot{\theta}_{i1} &= \sum_{j \in \mathcal{N}_i^{comm}} (\theta_{j1} - \theta_{i1}) - q_i^{-1} \omega_i \\ \dot{\theta}_{i2} &= S_2 \theta_{i2} + \sum_{j \in \mathcal{N}_i^{comm}} (\theta_{j2} - \theta_{i2}) - q_i^{-1} R^T \omega_i \\ u_i &= q_i^{-1} \theta_{i1} + q_i^{-1} R \theta_{i2}, \quad i = 1, 2, \dots, n, \end{aligned} \quad (23)$$

guarantee the solutions to the closed-loop system to converge asymptotically to the largest invariant set where  $\omega_i = 0$  for all  $i = 1, 2, \dots, n$ , and  $u = \bar{u}$ , with  $\bar{u}$  the optimal feedforward input.

## VII. SIMULATIONS

We illustrate the performance of the controllers on an academic example of the electricity grid. Consider a 4-area interconnected system<sup>3</sup>, as shown in Figure 1.

As a first example we assume the power demand  $P^l$  is constant and therefore the controller of Section V is applicable. The system is initially at steady state with a constant load  $P^l(t) = (3, 3, 3, 3)^T$ ,  $t \in [0, 20)$  and according to their cost functions generators take a different share in the power generation such that the total costs are minimized. At timestep 20 the load is changed to  $P^l(t) = (3, 3, 4, 5)^T$ ,  $t \geq 20$ . The frequency response to the control input is given in Figure 2. From Figure 2 we can see how the frequency drops due to the increased load. Furthermore we note that the controller regulates the power generation such that a new steady state condition is obtained where the frequency deviation is again zero and costs are minimized.

As a second example we consider a time-varying power demand as in (21). This time we rely on the controller proposed in Section VI with matrices  $S$  and  $R$  as in Example 1, with  $\mu = 0.1$ . The load profile is given by  $P^l(t) = (3, 3, 3, 3)^T + \sin(\frac{t}{10})(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4})^T$ ,  $t \in [0, 120)$  and  $P^l(t) = (3, 3, 4, 5)^T + \sin(\frac{t}{10})(2, 1\frac{2}{3}, \frac{1}{2})^T$ ,  $t \geq 120$ . From Figure 3 we can see how the controller provides a time-varying input such that the frequency deviation is driven to zero even in the presence of a time-varying load in such a way that the generation costs are minimized.

## VIII. CONCLUSIONS AND FUTURE WORK

The results presented in this paper enables a systematic approach to design a distributed controller for frequency regulation and cost minimization in power networks. Based

<sup>3</sup>This example network is taken from [2]. When applicable we use the same parameters for the generators and for the transmission lines.  $Q = \text{diag}(1, 2, 3, 4)$ .

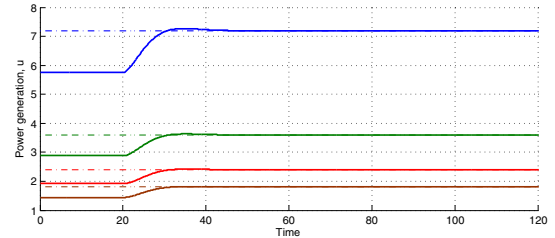
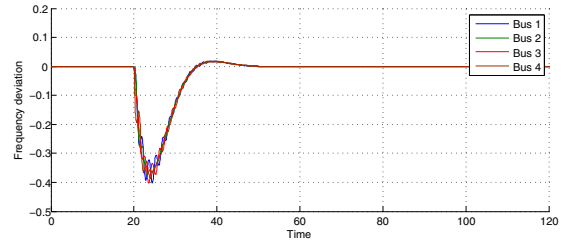


Fig. 2. Frequency response and control input using the controller of Section V. The constant load is increased at timestep 20, whereafter the frequency deviation is regulated back to zero and generation costs are minimized. The cost minimizing inputs for  $t \geq 20$  as in (11) are given by the dashed lines.

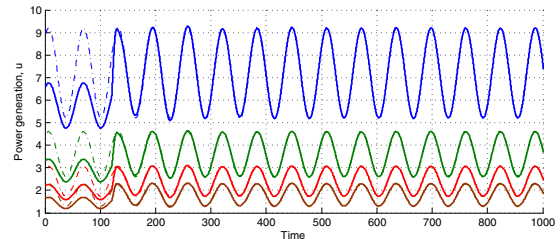
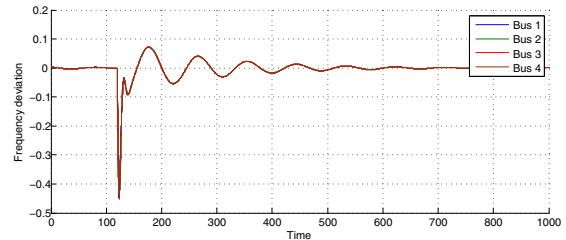


Fig. 3. Frequency response and control input using the controller of Section VI. The time-varying load is increased at timestep 120, whereafter the frequency is regulated back to zero and generation costs are minimized. The cost minimizing inputs for  $t \geq 120$  as in (11) are given by the dashed lines.

on the use of incremental passivity, internal models and optimal flow control problems we proposed a novel way to deal with the growing need for more advanced generation and load control in future power networks where the electricity production and demand becomes more volatile.

Future work will include the use of more accurate models of the power grid to model reactive power and voltage control. Based on results such as [11] larger classes of time-varying disturbances will be considered, by allowing the use of more general exosystems. Capacity constraints and non-quadratic cost functions will also be considered following the lines of [12]. Dynamic pricing of electricity enables the possibility to control demand. The presented approach here will be adapted to develop a pricing mechanism to use load-control as a

partly replacement of generation-control. The proposed controllers exchange information among them. Different controllers where such an exchange of information is not required should also be investigated. Finally the analysis has pointed out that asking for a zero frequency deviation in the presence of time-varying power demand restricts the power demand that can be dealt with. As a future research, we will investigate different problems of optimal frequency regulation where the frequency is allowed to differ from the zero frequency by small variations.

#### REFERENCES

- [1] I. Ibraheem, P. Kumar, and D. Kothari, "Recent philosophies of automatic generation control strategies in power systems," *Power Systems, IEEE Transactions on*, vol. 20, no. 1, pp. 346–357, 2005.
- [2] N. Li, L. Chen, C. Zhao, and L. Steven, "Connecting automatic generation control and economic dispatch from an optimization view," *Submitted to ACC*, 2014.
- [3] M. Andreasson, D. Dimarogonas, K. Johansson, and H. Sandberg, "Distributed vs. centralized power systems frequency control under unknown load changes," in *Proc. of European Control Conference*, Zurich, Switzerland, 2013.
- [4] M. Bürger and C. De Persis, "Internal models for nonlinear output agreement and optimal flow control," in *Proc. of 9th IFAC Symposium on Nonlinear Control Systems (NOLCOS)*, Toulouse, France, 2013, pp. 289 – 294.
- [5] —, "Dynamic coupling design for nonlinear output agreement and time-varying flow control," submitted for journal publication. arXiv:1311.7562 [cs.SY].
- [6] A. Pavlov and L. Marconi, "Incremental passivity and output regulation," *Systems and Control Letters*, vol. 57, pp. 400 – 409, 2008.
- [7] G.-B. Stan and R. Sepulchre, "Analysis of interconnected oscillators by dissipativity theory," *IEEE Transactions on Automatic Control*, vol. 52, no. 2, pp. 256 – 270, 2007.
- [8] C. De Persis, "Balancing time-varying demand-supply in distribution networks: an internal model approach," in *Proc. of European Control Conference*, Zurich, Switzerland, 2013, pp. 748 – 753.
- [9] J. Simpson-Porco, F. Dörfler, and F. Bullo, "Synchronization and power sharing for droop-controlled inverters in islanded microgrids," *Automatica*, vol. 49, no. 9, pp. 2603–2611, 2013.
- [10] F. Shaik, D. Zonetti, R. Ortega, J. Scherpen, and A. van der Schaft, "A port-Hamiltonian approach to power network modeling and analysis," *European Journal of Control*, vol. 56, 2013, to appear.
- [11] N. Cox, L. Marconi, and A. Teel, "Hybrid internal models for robust spline tracking," in *Proc. of 51st IEEE Conference on Decision and Control*, Maui, HI, 2012, pp. 4877 – 4882.
- [12] M. Bürger, C. De Persis, and F. Allgöwer, "Optimal pricing control in distribution networks with time-varying supply and demand," in *Proc. of the 21th International Symposium on Mathematical Theory of Networks and Systems*, Groningen, the Netherlands, 2014, arXiv:1403.6952 [cs.SY].
- [13] M. Bürger, C. De Persis, and S. Trip, "An internal model approach to (optimal) frequency regulation in power grids," arXiv:1403.7019 [cs.SY].
- [14] J. Machowski, J. Bialek, and J. Bumby, *Power System Dynamics: Stability and Control*, 2nd ed. Wiley, 2008.
- [15] J. Schiffer, R. Ortega, A. Astolfi, J. Raisch, and T. Sezi, "Synchronization of droop-controlled microgrids with distributed rotational and electronic generation," *Automatica*, 2014.
- [16] B. Jayawardhana, R. Ortega, E. Garcia-Canseco, and F. Castanos, "Passivity on nonlinear incremental systems: Application to PI stabilization of nonlinear RLC circuits," *Systems and Control Letters*, vol. 56, pp. 618 – 622, 2007.
- [17] M. Bürger, D. Zelazo, and F. Allgöwer, "Duality and network theory in passivity-based cooperative control," *Automatica*, 2013, submitted 2013. arXiv:1301.3676 [cs.SY].