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Combinatorial simplex algorithms can solve mean payoff games

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Abstract— **A combinatorial simplex algorithm is an instance of the simplex method in which the pivoting depends on combinatorial data only. We show that any algorithm of this kind admits a tropical analogue which can be used to solve mean payoff games. Moreover, any combinatorial simplex algorithm with a strongly polynomial complexity (the existence of such an algorithm is open) would provide in this way a strongly polynomial algorithm solving mean payoff games (all the arithmetic operations being performed on data polynomially bounded in the size of the input, in particular). Mean payoff games are known to be in NP $\cap$ co-NP; whether they can be solved in polynomial time is an open problem. Our algorithm relies on a tropical implementation of the simplex method over a real closed field of Hahn series. One of the key ingredients is a new scheme for symbolic perturbation which allows us to lift an arbitrary mean payoff game instance into a non-degenerate linear program over Hahn series.**

I. INTRODUCTION

The purpose of this paper is to establish a link between two notoriously open problems in theoretical computer science. It is unknown whether or not there is a strongly polynomial algorithm for linear programming. For mean payoff games, even the existence of a (not necessarily strongly) polynomial time algorithm to decide which player has a winning strategy is an open question. Without offering a solution to either problem, we show that a solution for the first problem, with additional properties, implies a solution for the second one. Our proof uses tropical geometry, or rather tropical linear algebra, in an essential way.

Mean payoff games were shown in [1] to be equivalent to feasibility problems in tropical linear programming. This raises the issue of tropicalizing classical linear programming algorithms. This problem is addressed in a recent work of the authors [3], which introduces a tropical analogue of the simplex method, and relates it to the classical simplex method over ordered fields. However, the algorithm of [3] is limited to primally and dually non-degenerate problems, and only provides a Phase II simplex method, that is, it requires a tropical basic point as additional input. Classically, Phase I, which finds a first basic point, can be reduced to a Phase II problem, but this requires to be able to deal with degenerate input. To overcome this obstacle we introduce a new scheme for symbolic perturbation which is tailored to the needs of tropical linear programming. This scheme is similar in spirit to other techniques known in computational geometry [4], [5] and linear optimization [7], [6]. In this way, we arrive at a method able to handle arbitrary tropical linear programming instances, and so, mean payoff games.

Of course, the performance of the tropical simplex algorithm depends on the choice of the pivoting rule. Here, we define a class of combinatorial pivoting rules, which can be described as follows. Classically, the objective function induces an orientation of the edges of the graph of the ordinary convex polyhedron described by the linear constraints. Phase II of the simplex method traces a path from a starting basic point to an optimum. A pivoting rule is combinatorial if the path is constructed incrementally using only the information available from the latter oriented graph.

The main result of this extended abstract (Theorem 6 and Corollary 7) says that a strongly polynomial classical simplex algorithm with a combinatorial pivoting rule satisfying mild conditions yields a strongly polynomial algorithm for solving mean payoff games. Detail of proofs can be found in the extended version of the present work [2].

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II. PRELIMINARIES

The tropical semiring usually refers to the max-plus semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$. More generally, a totally ordered abelian group $(G, +, \leqslant)$, gives rise to a tropical semiring $(\mathbb{T}(G), \oplus, \odot)$, where $\mathbb{T}(G) = G \cup \{\mathbb{0}_{\mathbb{T}(G)}\}$ is defined by adding a bottom element to G; the additive law $\oplus$ is the maximum for the order $\leqslant$; and the multiplicative law $\odot$ is the group addition $+$ extended to $\mathbb{T}(G)$ by setting $a \odot \mathbb{0}_{\mathbb{T}(G)} = \mathbb{0}_{\mathbb{T}(G)} \odot a = \mathbb{0}_{\mathbb{T}(G)}$ for all $a \in \mathbb{T}(G)$. These operations are extended to matrices in the usual way. When this is clear from the context, we simply write $\mathbb{T}$ for $\mathbb{T}(G)$ and $\mathbb{0}$ for $\mathbb{0}_{\mathbb{T}(G)}$.

A. *Mean payoff games and tropical linear programming*

A mean payoff game is described by two payment matrices $A, B \in \mathbb{T}(\mathbb{R})^{m\times n}$. The two players, called "Max" and "Min", alternatively play as follows. Given an initial state $\bar{\jmath} \in [n]$, Min chooses a state $i_1 \in [m]$ such that $A_{ij} \neq \mathbb{0}_{\mathbb{T}(\mathbb{R})}$ and pays $-A_{i_1\bar{\jmath}}$ to player Max. Then, Max chooses a state $j_1 \in [m]$ such that $B_{i_1 j_1} \neq \mathbb{0}_{\mathbb{T}(\mathbb{R})}$ and receives a payment $B_{i_1 j_1}$ from player Min. Then Min plays again and chooses a state $i_2 \in [m]$ as above. In this way, a play consists of an infinite sequence of states $\bar{\jmath}, i_1, j_1, i_2, j_2, \dots$. The mean payoff of Player Max for this sequence is

$$\liminf_{p\to\infty} \frac{-A_{i_1\bar{\jmath}} + B_{i_1 j_1} - A_{i_2 j_1} + \cdots + B_{i_p j_p}}{p} \enspace .$$

Under mild assumptions, this mean payoff is well-defined, and depends on the initial state $\bar{\jmath}$. The state $\bar{\jmath}$ is winning for Player Max if he can achieve a non-negative mean payoff, regardless of the actions chosen by player Min. Determining if an initial state is winning is equivalent to solving a tropical linear program.

Theorem 1 ([1])**.** *The initial state $\bar{\jmath}$ is winning for player Max if and only if the value of the following tropical polyhedron is not empty.*

$$x_{\bar{\jmath}} \geqslant 0, \quad A \odot x \leqslant B \odot x$$

As in the classical case, deciding feasibility of tropical polyhedra amounts to linear programming. Formally, a *tropical linear program* is a problem of the form:

$$\mathrm{LP}: \min_{x \in \mathbb{T}(G)^n} \{ c^\top \odot x \mid A^+ \odot x \oplus b^+ \geqslant A^- \odot x \oplus b^- \} \ , \quad (1)$$

where $A^+, A^- \in \mathbb{T}(G)^{m \times n}$, $b^+, b^- \in \mathbb{T}(G)^m$ and $c \in \mathbb{T}(G)^n$.

B. Linear programming over Hahn series

A tropical semiring $\mathbb{T}(G)$ arises naturally from the value group of the field of (real) Hahn series $\mathbb{R}[\![t^G]\!]$. This field consists of formal power series $\boldsymbol{x} := \sum_{\alpha \in \Lambda} x_\alpha t^\alpha$ where the *support* $\Lambda \subset G$ is well-ordered, and the coefficients x_α are non-zero real numbers. The *valuation* of $\boldsymbol{x} \in \mathbb{R}[\![t^G]\!]$ is $\mathrm{val}(\boldsymbol{x}) = -\min\{\alpha \in \Lambda\}$ with the convention $-\min(\emptyset) = \mathbb{0}_{\mathbb{T}(G)}$. The series $\boldsymbol{x} \neq 0$ is positive, and we write $\boldsymbol{x} > 0$, when the leading coefficient $x_{-\mathrm{val}(\boldsymbol{x})}$ is positive. Through this relation, $\mathbb{R}[\![t^G]\!]$ is an ordered field.

The valuation forms an order-preserving homomorphism from the semiring of non-negative Hahn series to $\mathbb{T}(G)$. This entails the following relation between linear programs over Hahn series and tropical linear programs.

Proposition 2 ([3, Prop. 7])**.** *For any tropical linear program LP of the form* (1)*, there exists $\boldsymbol{A}^\pm \in \mathrm{val}^{-1}(A^\pm)$, $\boldsymbol{b}^\pm \in \mathrm{val}^{-1}(b^\pm)$ and $\boldsymbol{c} \in \mathrm{val}^{-1}(c)$ such that the linear program over $\mathbb{R}[\![t^G]\!]$*

$$\boldsymbol{LP}: \min_{\boldsymbol{x} \in (\mathbb{R}[\![t^G]\!])^n} \{ \boldsymbol{c}^\top \boldsymbol{x} \mid \boldsymbol{A}^+ \boldsymbol{x} + \boldsymbol{b}^+ \geqslant \boldsymbol{A}^- \boldsymbol{x} + \boldsymbol{b}^- \text{ and } \boldsymbol{x} \geqslant 0 \},$$

satisfies the following properties.

- *The image under the valuation map of the feasible set of $\boldsymbol{LP}$ is equal to the feasible set of LP. In particular, $\boldsymbol{LP}$ is feasible if, and only if, LP is feasible.*
- *If $\boldsymbol{LP}$ admits $\boldsymbol{x}^*$ as an optimal solution, then $\mathrm{val}(\boldsymbol{x}^*)$ is an optimal solution of LP.*

C. Classical simplex method

As $\mathbb{R}[\![t^G]\!]$ is an ordered field, common results related to linear programming hold (Farkas' lemma, duality, *etc*). In particular, the simplex method can be used to solve linear programs over Hahn series.

We recall basic facts about the simplex method. We restrict our exposition to linear programs $\boldsymbol{LP}$ which are non-degenerate, and whose feasible sets are bounded and included in the interior of the positive orthant. In this setting, a *basis* is a subset $I \subset [m] = \{1, \dots, m\}$ of cardinality n such that $\det(\boldsymbol{A}_I^+ - \boldsymbol{A}_I^-) \neq 0$. A basis I defines a *(feasible) basic point* if the unique solution of the system $\boldsymbol{A}_I^+ \boldsymbol{x} + \boldsymbol{b}_I^+ = \boldsymbol{A}_I^- \boldsymbol{x} + \boldsymbol{b}_I^-$ is a feasible point of $\boldsymbol{LP}$. In this case, the latter point is denoted by $\boldsymbol{x}^I$. An *edge* $\boldsymbol{\mathcal{E}}_K$ is defined as a line segment between two adjacent basic points $x^{K \cup \{i\}}$ and $x^{K \cup \{i'\}}$, where $|K| = n-1$, $i, i' \notin K$, and $i \neq i'$. Every edge consists of the feasible points of $\boldsymbol{LP}$ which also satisfy $\boldsymbol{A}_K^+ \boldsymbol{x} + \boldsymbol{b}_K^+ = \boldsymbol{A}_K^- \boldsymbol{x} + \boldsymbol{b}_K^-$.

Algorithm 1: Simplex method

Input: The oracles $\mathsf{Pivot}_{\boldsymbol{LP}}, \mathsf{SignRedCosts}_{\boldsymbol{LP}}$, a pivoting strategy $\phi_{\boldsymbol{LP}}$, and a basis I^1 of a feasible basic point of $\boldsymbol{LP}$.
Output: A basis I^{opt} of an optimal basic point of $\boldsymbol{LP}$

$k \leftarrow 1$
while $\mathsf{SignRedCosts}_{\boldsymbol{LP}}(I^k)$ has a negative entry **do**
  $i^{\mathsf{out}} = \phi_{\boldsymbol{LP}}(\{I^1, \dots, I^k\})$
  $I^{k+1} \leftarrow \mathsf{Pivot}_{\boldsymbol{LP}}(I^k, i^{\mathsf{out}})$
  $k \leftarrow k+1$
return I^k

The principle of the simplex algorithm is to iterate two elementary steps, namely pivoting and computing reduced costs. Given a basic point $\boldsymbol{x}^I$ and $i^{\mathsf{out}} \in I$, the *pivoting* step consists in going along the incident edge $\boldsymbol{\mathcal{E}}_{I \setminus \{i^{\mathsf{out}}\}}$, up to another basic point $\boldsymbol{x}^{I'}$. The new basis I' is of the form $I \setminus \{i^{\mathsf{out}}\} \cup \{i^{\mathsf{ent}}\}$ for some $i^{\mathsf{ent}} \notin I$, and since the linear program is non-degenerate, i^{ent} is uniquely defined. We consequently denote by $\mathsf{Pivot}_{\boldsymbol{LP}}(I, i^{\mathsf{out}})$ a routine which returns the basis $I \setminus \{i^{\mathsf{out}}\} \cup \{i^{\mathsf{ent}}\}$.

The *reduced costs* associated with the basis I refer to the components of the unique solution $\boldsymbol{y}^I$ of the system $(\boldsymbol{A}_I^+ - \boldsymbol{A}_I^-)^\top \boldsymbol{y} = \boldsymbol{c}$. Moving from the basic point $\boldsymbol{x}^I$ along the edge $\boldsymbol{\mathcal{E}}_K$ for $K = I \setminus \{i\}$ improves the objective function if, and only if, the reduced cost $\boldsymbol{y}_i^I$ is negative. We denote by $\mathsf{SignRedCosts}_{\boldsymbol{LP}}(I)$ a function which returns the signs of the entries of $\boldsymbol{y}^I$.

Given an initial basis I_1, corresponding to a feasible basic point of $\boldsymbol{LP}$, the simplex method builds a sequence of bases $I_1, I_2, \dots, I_N$, where the last basis I^N yields an optimal basic point. This sequence satisfies $I_{k+1} = \mathsf{Pivot}_{\boldsymbol{LP}}(I_k, i_k^{\mathsf{out}})$ for all $k \in [N-1]$. At every iteration, the leaving index i_k^{out} is chosen by a *pivoting strategy* $\phi_{\boldsymbol{LP}}$ which takes as input $(I_1, \dots, I_k)$ the history up to time k. The algorithm terminates when all the signs returned by $\mathsf{SignRedCosts}_{\boldsymbol{LP}}$ are non-negative.

D. Tropical simplex method

The tropical simplex algorithm, introduced in [3], has the same structure as the classical simplex algorithm. It is defined by replacing the functions $\mathsf{Pivot}_{\boldsymbol{LP}}$ and $\mathsf{SignRedCosts}_{\boldsymbol{LP}}$ by their tropical analogues. They respectively allow to pivot along a tropical edge, and to compute the signs of the tropical reduced costs. However, the tropical simplex algorithm is restricted to linear programs satisfying the following two assumptions:

Assumption A (Finiteness)**.** *The feasible set of LP is a bounded subset of G^n.*

Assumption B (Genericity)**.** *The matrix $\begin{pmatrix} A^+ \oplus A^- & b^+ \oplus b^- \\ c & \mathbb{0} \end{pmatrix}$ is tropically generic.*

Recall that a matrix M is said to be *tropically generic* if, for every square submatrix W of M, the tropical permanent of W, given by:

$$\operatorname{tper} W = \max_{\sigma \in \operatorname{Sym}([n])} W_{1\sigma(1)} + \cdots + W_{n\sigma(n)} \ , \qquad (2)$$

is either equal to $\mathbb{0}$, or if the maximum in (2) is attained only once.

The main result of [3] is the following:

Theorem 3 ([3, Th. 1])**.** *Under Assumptions A and B, the tropical simplex algorithm terminates and returns an optimal solution of LP for any tropical pivoting rule. Every iteration (pivoting and computing reduced costs) can be done in $O(n(m+n))$ arithmetic operations over $\mathbb{T}$, and in linear space. Moreover, the algorithm traces the image under the valuation map of the path followed by the classical simplex algorithm applied to any lift $\boldsymbol{LP}$, with a compatible pivoting rule.*

To fully tropicalize the simplex algorithm it remains to: 1) handle tropical linear programs which do not satisfy Assumptions A and B; 2) identify pivoting strategies $\phi_{\boldsymbol{LP}}$ which admit a tropical implementation.

III. Perturbation/compactification of tropical polyhedra

We outline how to apply the tropical simplex method to arbitrary tropical linear programs. More details can be found in [2], where, *e.g.*, Phase I of the tropical simplex method is discussed. This yields an algorithm to determine whether the tropical linear program is feasible, and to find an initial feasible basis (if any).

A. Compactification

We show that an arbitrary tropical linear program LP defined in $\mathbb{T}(G)$ is equivalent to a tropical linear program $\overline{\mathsf{LP}}$ which satisfies Assumption A. The problem $\overline{\mathsf{LP}}$ is defined in the semiring $\mathbb{I} = \mathbb{T}(\mathbb{R} \times G)$, where the group $\mathbb{R} \times G$ is ordered lexicographically. Intuitively, a pair $(\alpha, \beta) \in \mathbb{R} \times G$ represents a scalar $\alpha M + \beta$, where M is sufficiently large. Thus pairs of the form $(\alpha, \cdot)$ with $\alpha < 0$ correspond to the bottom element $\mathbb{0}_{\mathbb{T}(G)}$ of $\mathbb{T}(G)$, while $(0, \beta)$ encodes the element $\beta \in G$.

The feasible points x of $\overline{\mathsf{LP}}$ are enclosed in a box $\mathsf{u}_j \geqslant \mathsf{x}_j \geqslant \mathsf{l}_j$, where the u_j are of the form $(\alpha, \cdot)$ with $\alpha > 0$ and $\mathsf{l}_j = (\alpha, \cdot)$ with $\alpha < 0$. Then, it can be shown that the feasible points of $\overline{\mathsf{LP}}$ which also belong to $\{(\alpha, \cdot) \mid \alpha \leqslant 0\}^n$ correspond to feasible points of LP, while the remaining feasible points of $\overline{\mathsf{LP}}$ are in correspondence with the rays of the recession cone of the feasible set of LP.

B. Perturbation

Embedding $\overline{\mathsf{LP}}$ in a yet bigger semiring yields an equivalent problem $\widetilde{\mathsf{LP}}$ which satisfies Assumption B. The problem $\widetilde{\mathsf{LP}}$ is defined in the semiring $\mathbb{G} = \mathbb{T}(\mathbb{R} \times G \times H)$, where H is a totally ordered group, and $\mathbb{R} \times G \times H$ is ordered lexicographically.

A tuple $(\alpha, \beta; \gamma) \in \mathbb{R} \times G \times H$ represents a pair $(\alpha, \beta) + \gamma\epsilon$, where ϵ is an infinitely small formal value. Thus γ plays the role of a symbolic perturbation of (α, β).

Consider an arbitrary matrix $M = (M_{ij}) \in \mathbb{I}^{m \times n}$, and let us instantiate $H = \mathbb{R}^m$. Let $E = (\epsilon_{ij}) \in H^{m \times n}$ be the matrix whose (i,j)-entry is the vector $j\delta^i$, where δ^i is the i-th element of the canonical basis of $\mathbb{R}^m$. We define the perturbed matrix $M[E] = (\widetilde{\mathsf{M}}_{ij}) \in \mathbb{G}^{m \times n}$ by $\widetilde{\mathsf{M}}_{ij} = (M_{ij}, \epsilon_{ij})$ if $M_{ij} \neq \mathbb{0}_{\mathbb{I}}$ and $\widetilde{\mathsf{M}}_{ij} = \mathbb{0}_{\mathbb{G}}$ otherwise.

Proposition 4. *The matrix $M[E]$ is tropically generic.*

The problem $\widetilde{\mathsf{LP}}$ is obtained by perturbing the matrix of the linear program $\overline{\mathsf{LP}}$ as above. In this way, $\widetilde{\mathsf{LP}}$ satisfies both Assumptions A and B. Furthermore, solving $\widetilde{\mathsf{LP}}$ yields a solution to $\overline{\mathsf{LP}}$ and thus of LP.

IV. Combinatorial pivoting rules can be tropicalized

We consider pivoting strategies $\phi_{\boldsymbol{LP}}$ which can be described by Turing machines with oracles, the oracles being parameterized by the problem at hand $\boldsymbol{LP}$. At step k, the strategy $\phi_{\boldsymbol{LP}}$ takes as only input the history of bases $(I_1, \ldots, I_k)$. Thus, the Turing machine can only make operations on the input $(I_1, \ldots, I_k)$. The relevant information on the problem $\boldsymbol{LP}$ is given through the oracles. For example, Bland's rule, which selects the improving edge with smallest index, can be seen as a Turing machine with oracles giving the sign of the reduced costs.

We say that a strategy $\phi_{\boldsymbol{LP}}$ is *combinatorial* if every oracle it uses returns the sign of a determinant of a submatrix of $\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A}^+ - \boldsymbol{A}^- & \boldsymbol{b}^+ - \boldsymbol{b}^- \\ \boldsymbol{c} & 0 \end{pmatrix}$, the matrix encoding the problem $\boldsymbol{LP}$. For instance, Bland's rule, as well as certain lookahead rules exploring a bounded neighborhood of the current basic point in the graph of the polyhedron, are combinatorial. We now show how to tropicalize these oracles. Thus, any combinatorial pivoting strategy admits a tropical implementation.

Let $\boldsymbol{W}$ be a $k \times k$ square submatrix of $\boldsymbol{M}$. We want to evaluate the sign of

$$\det \boldsymbol{W} = \sum_{\sigma} \boldsymbol{W}_{1\sigma(1)} \boldsymbol{W}_{2\sigma(2)} \ldots \boldsymbol{W}_{k\sigma(k)} \ .$$

If there is a unique monomial $\boldsymbol{W}_{1\sigma(1)} \boldsymbol{W}_{2\sigma(2)} \ldots \boldsymbol{W}_{k\sigma(k)}$ with maximal valuation among the latter sum, then the sign of $\det \boldsymbol{W}$ is given by the sign of this monomial. This always happens with the matrices we consider. Indeed, we apply the tropical simplex method to tropical linear programs which are perturbed as described in section III-B. Thus, the matrix $\boldsymbol{M}$ describing the Hahn linear programs always have a valuation of the form $M[E]$. Hence, by Proposition 4, the submatrices of $M[E]$ are always tropically non-singular, or with a $\mathbb{0}$ tropical permanent.

As a consequence, we can compute the sign of $\det \boldsymbol{W}$ by applying the Hungarian method to either obtain the unique maximizing permutation σ^* in $\mathrm{tper}(\mathrm{val}(\boldsymbol{W}))$, or to determine that $\mathrm{tper}(\mathrm{val}(\boldsymbol{W})) = \mathbb{0}$. In the former case, the signs of the entries $\boldsymbol{W}_{1\sigma^*(1)}, \dots, \boldsymbol{W}_{k\sigma^*(k)}$ yield the sign of $\mathrm{tdet}\, \boldsymbol{W}$. In the latter case, $\mathrm{tdet}\, \boldsymbol{W} = 0$. This can be done in strongly polynomial time since the Hungarian method has a strongly polynomial complexity, see [10, §17.3].

V. Combinatorial pivoting rules applied to mean payoff games

We now compare the number of iterations of the tropical simplex method with the number of iterations of the classical simplex method on real numbers. Let K be an ordered field. We are interested in the cases where $K = \mathbb{R}$, or $K = \mathbb{R}[\![t^G]\!]$ where G is an divisible totally ordered group. In these two cases, K is a real-closed field, see [9] and [8, (6.11) p. 143]. Tarski's principle states that the first-order theory of real-closed field is model-complete. We will use this to compare the behavior of the simplex method on Hahn series and on real numbers. Let $N_K(n, m, \phi)$ be the maximal length of a run of the simplex method, equipped with the strategy ϕ, on linear program with n variables and m constraints on the field K.

Proposition 5. *Assume that G is a divisible totally ordered group and ϕ a combinatorial pivoting strategy. Then,*

$$N_{\mathbb{R}}(n, m, \phi) = N_{\mathbb{R}[\![t^G]\!]}(n, m, \phi) \ .$$

To prove the proposition, it suffices to verify [2] that, given integers n, m and N, the following statement is a first-order sentence in the language of ordered fields: "for any matrices $\boldsymbol{A} \in K^{m\times n}, \boldsymbol{b} \in K^m$ and $\boldsymbol{c} \in K^n$, the simplex method equipped with the pivoting strategy ϕ solves the linear program

$$\min\{\boldsymbol{c}^\top \boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b} \geqslant 0, \boldsymbol{x} \geqslant 0, \boldsymbol{x} \in K^n\}$$

in at most N iterations, starting from any initial basis."

Theorem 6. *Let ϕ be a combinatorial pivoting strategy such that the following conditions are satisfied:*

- *the number of arithmetic operations and calls to the oracles performed by ϕ is polynomially bounded by n and m;*
- *the space complexity of ϕ is polynomially bounded by n and m.*

Suppose that the classical simplex algorithm equipped with ϕ is strongly polynomial on all linear programs over $\mathbb{R}$. Then, all tropical linear programs can be solved in strongly polynomial time. In particular, mean payoff games can be solved in strongly polynomial time.

Proof. If the simplex algorithm is strongly polynomial over $\mathbb{R}$, then $N_{\mathbb{R}}(n, m, \phi)$ is a polynomial in n and m. By Theorem 3 and Proposition 5, the number of iterations of the tropical simplex algorithm is smaller than $N_{\mathbb{R}}(n, m, \phi)$. By assumptions on the pivoting strategy, the tropical simplex algorithm is strongly polynomial. □

The following result is an immediate corollary of Theorems 1 and 6.

Corollary 7. *Let ϕ be a combinatorial pivoting strategy as in Theorem 6. Suppose that the classical simplex algorithm equipped with ϕ is strongly polynomial on all non-degenerate linear programs over $\mathbb{R}$. Then, mean payoff games can be solved by a strongly polynomial algorithm.*

References

[1] M. Akian, S. Gaubert, and A. Guterman. Tropical polyhedra are equivalent to mean payoff games. *International Journal of Algebra and Computation*, 22(1):125001, 2012.
[2] X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig. Combinatorial simplex algorithms can solve mean payoff games. Submitted. E-print arXiv:1309.5925, 2013.
[3] X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig. Tropicalizing the simplex algorithm. Submitted. E-print arXiv:1308.0454, 2013.
[4] H. Edelsbrunner. *Algorithms in combinatorial geometry*, volume 10 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Berlin, 1987.
[5] I.Z. Emiris, J.F. Canny, and R. Seidel. Efficient perturbations for handling geometric degeneracies. *Algorithmica*, 19(1-2):219–242, 1997.
[6] J.A. Filar, E. Altman, and K.E. Avrachenkov. An asymptotic simplex method for singularly perturbed linear programs. *Oper. Res. Lett.*, 30(5):295–307, 2002.
[7] R.G. Jeroslow. Asymptotic linear programming. *Operations Research*, 21(5):1128–1141, 1973.
[8] P. Ribenboim. Fields: algebraically closed and others. *Manuscripta Math.*, 75:115–150, 1992.
[9] A. Robinson. *Complete theories*. North Holland, 1977.
[10] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Paths, flows, matchings, Chapters 1–38.