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On the interconnections of linear passive systems and maximal monotone set-valued mappings

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Abstract— **This paper deals with a class of dynamical systems obtained from interconnecting linear systems with static set-valued relations. We first show that such an interconnection can be described by a differential inclusions with a maximal monotone set-valued mappings when the underlying linear system is passive and the static relation is maximal monotone. Based on the classical results on such differential inclusions, we conclude that such interconnections are well-posed in the sense of existence and uniqueness of solutions.**

I. Introduction

Variational inequalities were introduced by Stampacchia in 1964 [26] as a tool in the study of elliptic partial differential equations, and have since been recognized as instrumental in a large class of optimization and equilibrium problems. Applications range from elastoplasticity to traffic and from electrical networks to mathematical finance; see for instance [12], [17]. The role of maximal monotonicity in the context of variational inequalities, as a sufficient condition for well-behavedness, can be compared to the role of convexity in optimization problems. Maximal monotone mappings were introduced in 1961 by Minty [21], who had already earlier applied the notion of monotone relations in an abstract formulation for electrical networks of nonlinear resistors [20]. Extensions to dynamic problems were undertaken in the same decade; intimate connections between semigroups of nonlinear contractions and maximal monotone mappings were established by Crandall and Pazy [11] and further developed by Brézis [3].

The development of the theory of semigroups of nonlinear contractions took place in the classical context of dynamics given by a closed system of (partial) differential equations. Engineers have long appreciated the power of *open* (input-output) dynamical systems as a device for modeling as well as for analysis. It comes naturally in many applications in the engineering sciences, as well as in biology and economics, to look at a dynamical system as a composite of smaller systems which are connected by the specification of relations between certain variables associated to the subsystems. These variables may be referred to as "inputs" and "outputs", or more generally as "connecting variables" since the suggestion of unidirectionality that comes with the input/output terminology is not always appropriate. Systems equipped with connecting variables in this sense may be simply referred to as "open dynamical systems". Early contributions were made in the 1930's in the field of electrical engineering by among others Nyquist and Bode, and the field has received intensive study ever since the pioneering work of Kalman around 1960 and the associated successes in the Apollo space program and in many other applications.

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Within the class of open dynamical systems, linear time-invariant systems play a special role as a prime example and as a first breeding ground of ideas that are later developed in wider contexts. More or less similarly, linear complementarity problems [10] take a special position among variational inequalities. Dynamical systems that arise as interconnections of linear time-invariant systems and linear complementarity problems came under investigation in the 1990's under the name "linear complementarity systems" [15], [28]. Part of the motivation came from the fact that these systems can be looked at as a particular class of systems with mixed continuous and discrete state variables, also called "multimodal systems" or "hybrid systems". More generally, differential variational inequalities were studied by Pang and Stewart [23]. Linear time-invariant systems together with static relations described by set-valued mappings have been used extensively. An incomplete inventory includes electrical networks with switching elements as in power converters [8], [14], [29], linear relay systems [1], [19], piecewise linear systems [9], and projected dynamical systems [16], [22]; see also [25], [27] for further examples.

The history of linear time-invariant systems connected to static (nonlinear) relations in fact goes back a long way. This way of describing a dynamical system has been used intensively as a tool in stability analysis within the context of so-called Lur'e systems; see [18] for a survey. The notion of passivity (also known as dissipativity) plays an important role in this theory. The term is used here as a description of a characteristic of an open dynamical system, and is motivated by the notion of stored energy in electrical networks and in many other applications in physics. The term "dissipativity" is used as well in the context of maximal monotone mappings; in fact, in their paper cited above [11], Crandall and Pazy use the term "dissipative set" in place of "maximal monotone mapping". This already indicates that there are strong conceptual relations between the notions of passivity and maximal monotonicity. Indeed, passive complementarity systems present themselves as a natural class of dynamical systems [7].

In this paper, our goal is to establish the well-posedness (in the sense of existence and uniqueness of solutions) for systems that arise as interconnections of passive linear time-

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invariant systems and maximal monotone mappings. Our proof strategy relies on a reduction to the classical case of a closed dynamical system. To achieve this, we present a new result in the spirit of preservation of maximal monotonicity under certain operations. Such results are known to be often nontrivial; even the question whether the sum of two maximal monotone mappings is again maximal monotone does not have a straightforward answer (cf. [24, Section 12.F]). The well-posedness of interconnections of linear passive systems with maximal monotone mappings has been studied before by Brogliato [4]. In the cited paper, well-posedness is proved under some additional conditions, which were later partly removed in [6]. Here we obtain the result without imposing additional conditions.

II. Preliminaries

A. Maximal monotone set-valued mappings

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued mapping, that is $F(x) \subseteq \mathbb{R}^n$ for each $x \in \mathbb{R}^n$. We define its domain, image, and graph, respectively, as follows:

$$\begin{aligned} \operatorname{dom}(F) &= \{x \mid F(x) \neq \varnothing\} \\ \operatorname{im}(F) &= \{y \mid \text{there exists } x \text{ such that } y \in F(x)\} \\ \operatorname{graph}(F) &= \{(x, y) \mid y \in F(x)\}. \end{aligned}$$

The inverse mapping $F^{-1} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by $F^{-1}(y) = \{x \mid y \in F(x)\}$.

Throughout the paper, we are interested in the so-called maximal monotone set-valued mappings. A set valued-mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is said to be *monotone* if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geqslant 0 \tag{1}$$

for all $(x_i, y_i) \in \operatorname{graph}(F)$. It is said to be *maximal monotone* if there is no other monotone set-valued mapping whose graph strictly contains that of F. We refer to [3] and [24] for detailed treatment of maximal monotone mappings.

B. Linear passive systems

A linear system $\Sigma(A, B, C, D)$

$$\dot{x}(t) = Ax(t) + Bz(t) \tag{2a}$$

$$w(t) = Cx(t) + Dz(t) \tag{2b}$$

is said to be *passive*, if there exists a nonnegative-valued *storage function* $V : \mathbb{R}^n \to \mathbb{R}_+$ such that the dissipation inequality

$$V(x(t_1)) + \int_{t_1}^{t_2} z^T(\tau) w(\tau)\, d\tau \leqslant V(x(t_2)) \tag{3}$$

is satisfied for all $0 \leqslant t_1 \leqslant t_2$ and for all trajectories $(z, x, w) \in L_{2,\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m) \times AC_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times L_{2,\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$ of the system (2).

The classical Kalman-Yakubovich-Popov lemma states that the system (2) is passive if, and only if, the linear matrix inequalities

$$K = K^T \geqslant 0 \qquad \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D^T + D) \end{bmatrix} \leqslant 0 \tag{4}$$

admits a solution K. Moreover, $V(x) = \frac{1}{2} x^T K x$ defines a storage function in case K is a solution the linear matrix inequalities (4).

III. Linear systems coupled to relations

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bz(t) + u(t) \tag{5a}$$

$$w(t) = Cx(t) + Dz(t) \tag{5b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^n$ is the input, and $(z, w) \in \mathbb{R}^{m+m}$ are the external variables that satisfy

$$\big(-z(t), w(t)\big) \in \operatorname{graph}(M) \tag{5c}$$

for some set-valued map $M : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$.

By solving z from the relations (5b) and (5c), we obtain the differential inclusion

$$\dot{x}(t) \in -H(x(t)) + u(t) \tag{6}$$

where

$$H(x) = -Ax + B(M + D)^{-1}(Cx) \tag{7}$$

and

$$\operatorname{dom}(H) = C^{-1}\big(\operatorname{im}(M + D)\big). \tag{8}$$

IV. Main results

Maximal monotonicity of the set-valued mapping H as defined in (7) will play a key role in our development. The following theorem asserts that H is maximal monotone if the underlying linear system is passive and the set-valued mapping M is maximal monotone.

Theorem 1 *Suppose that*

i. $\Sigma(A, B, C, D)$ *is passive with the storage function* $x \mapsto \frac{1}{2} x^T x$*,*
ii. M *is maximal monotone, and*
iii. $\operatorname{im} C$ *meets* $\operatorname{rint}(\operatorname{im}(M + D))$.

Then, the set-valued mapping H *defined in* (7) *is maximal monotone.*

Well-posedness of systems of the form (5) and their variants has been addressed in several papers [2], [4]–[6], [13] for linear passive (or passive-like) systems and maximal monotone mappings. However, the relevant results appeared in these papers require extra conditions on the linear system and/or the maximal monotone mapping. The following theorem provides conditions for the existence and uniqueness of solutions to the differential inclusion (6) when the linear system $\Sigma(A, B, C, D)$ is passive and the set-valued map M is maximal monotone without requiring any additional conditions.

Theorem 2 *Suppose that*

i. $\Sigma(A, B, C, D)$ *is passive with the storage function* $x \mapsto \frac{1}{2} x^T K x$ *where* K *is positive definite,*
ii. M *is maximal monotone, and*
iii. $\operatorname{im} C$ *meets* $\operatorname{rint}(\operatorname{im}(M + D))$.

Then, for each initial condition x_0 *such that* $Cx_0 \in \operatorname{cl}(\operatorname{im}(M + D))$ *and locally integrable function* u*, the differential inclusion* (6) *admits a unique solution.*

V. Concluding remarks

In this paper, we have shown that the interconnection of a linear system with a static set-valued relation is well-posed in the sense of existence and uniqueness of solutions whenever the underlying linear system is passive and the static relation is maximal monotone. Similar well-posedness results have already appeared in the literature with extra conditions on the linear systems as well as the static relations. Removing those extra conditions requires employing a completely different set of arguments (and hence tools). Based on the recent characterisations of maximal monotonicity, we have shown that such interconnections can be represented by differential inclusions with maximal monotone set-valued mappings. As such, the classical well-posedness results for such differential inclusions can be immediately applied to the class of systems at the hand.

References

[1] A. Y. Pogromsky, W. P. M. H. Heemels, and H. Nijmeijer. On solution concepts and well-posedness of linear relay systems. *Automatica*, 39:2139–2147, 2003.

[2] S. Adly and D. Goeleven. A stability theory for second-order nonsmooth dynamical systems with application to friction problems. *Journal de Mathematiques Pures et Appliquees*, 83(1):17–51, 2004.

[3] H. Brézis. *Operateurs Maximaux Monotones*. North-Holland, 1973.

[4] B. Brogliato. Absolute stability and the Lagrange-Dirichlet theorem with monotone multivalued mappings. *Systems & Control Letters*, 51:343–353, 2004.

[5] B. Brogliato and D. Goeleven. The Krakovskii-LaSalle invariance principle for a class of unilateral dynamical systems. *Mathematics of Control, Signals and Systems*, 17(1):57–76, 2005.

[6] B. Brogliato and D. Goeleven. Well-posedness, stability and invariance results for a class of multivalued lurÕe dynamical systems. *Nonlinear Analysis: Theory, Methods & Applications*, 74(1):195–212, 2011.

[7] M. K. Camlibel, W. P. M. H. Heemels, and J. M. Schumacher. On linear passive complementarity systems. *European Journal of Control*, 8(3):220–237, 2002.

[8] M. K. Camlibel, W. P. M. H. Heemels, A. J. van der Schaft, and J. M. Schumacher. Switched networks and complementarity. *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, 50(8):1036–1046, 2003.

[9] M. K. Camlibel and J. M. Schumacher. Existence and uniqueness of solutions for a class of piecewise linear dynamical systems. *Linear Algebra and its Applications*, 351–352:147–184, 2002.

[10] R.W. Cottle, J.-S. Pang, and R.E. Stone. *The linear complementarity problem*. Academic Press, Boston, 1992.

[11] M.G. Crandall and A. Pazy. Semi-groups of nonlinear contractions and dissipative sets. *Journal of Functional Analysis*, 3:376–418, 1969.

[12] F. Facchinei and J.S. Pang. *Finite-dimensional variational inequalities and complementarity problems*. Springer, New York, 2003.

[13] D. Goeleven and B. Brogliato. Stability and instability matrices for linear evolution variational inequalities. *IEEE Transactions on Automatic Control*, 49(4):521–534, 2004.

[14] W. P. M. H. Heemels, M. K. Camlibel, and J. M. Schumacher. On the dynamic analysis of piecewise-linear networks. *IEEE Transactions on Circuits and Systems–I: Fundamental Theory and Applications*, 49(3):315–327, March 2002.

[15] W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland. Linear complementarity systems. *SIAM J. Appl. Math.*, 60(4):1234–1269, 2000.

[16] W.P.M.H. Heemels, J.M. Schumacher, and S. Weiland. Projected dynamical systems in a complementarity framework. *Operations Research Letters*, 27:83–91, 2000.

[17] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*. Academic Press, New York, 1980.

[18] M. R. Liberzon. Essays on the absolute stability theory. *Automation and Remote Control*, 67:1610–1644, 2006.

[19] Y. J. Lootsma, A. J. van der Schaft, and M. K. Camlibel. Uniqueness of solutions of relay systems. *Automatica*, 35(3):467–478, 1999.

[20] G.J. Minty. Monotone networks. *Proceedings of the Royal Society of London, Series A*, 257:194–212, 1960.

[21] G.J. Minty. On the maximal domain of a “monotone” function. *Michigan Mathematical Journal*, 8:135–137, 1961.

[22] A. Nagurney and D. Zhang. *Projected dynamical systems and variational inequalities with applications*. Springer, New York, 1995.

[23] J.S. Pang and D.E. Stewart. Differential variational inequalities. *Mathematical Programming*, pages 345–424, 2008.

[24] R. T. Rockafellar and J. B. Wets. *Variational Analysis*. A Series of Comprehensive Studies in Mathematics 317. Springer, 1998.

[25] J. M. Schumacher. Complementarity systems in optimization. *Mathematical Programming Series B*, 101:263–295, 2004.

[26] G. Stampacchia. Formes bilineaires coercitives sur les ensembles convexes. *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, 258:4413–4416, 1964.

[27] D.E. Stewart. *Dynamics with inequalities. Impacts and hard constraints*. SIAM, Philadelphia, 2011.

[28] A. J. van der Schaft and J. M. Schumacher. Complementarity modelling of hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):483–490, 1998.

[29] F. Vasca, L. Iannelli, M. K. Camlibel, and R. Frasca. A new perspective for modeling power electronics converters: complementarity framework. *IEEE Transactions on Power Electronics*, 24(2):456–468, 2009.