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On the reachability of discrete-time bimodal piecewise linear systems

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***Abstract*— This paper studies reachability problem for discrete-time bimodal piecewise linear systems. Lack of convexity in discrete-time reachability problem is a major issue that prevents generalizations of the existing results for such systems in continuous-time. In [7], necessary and sufficient conditions for controllability of discrete-time bimodal systems were presented only under a quite restrictive condition on the dimension of the underlying zero dynamics. In this paper, we take a completely different approach that is based on the recent results for the reachability of linear systems with conical output constraints and provide necessary and sufficient conditions without any assumptions on the zero dynamics.**

I. INTRODUCTION

Apart from linear systems, bimodal systems are the simplest instances of piecewise affine dynamical systems. In the continuous-time case, algebraic necessary and sufficient conditions for controllability of bimodal systems has been presented in [2] and for stabilizability in [4]. Based on the ideas and approach of [2] , controllability of conewise linear systems was studied in [3], complementarity systems in [1], and finally continuous piecewise affine dynamical systems in [6].

The core idea of this line of research was to reduce the controllability problem of a piecewise linear system to the controllability problem of a corresponding linear system with input constraints.

Adopting a similar approach for discrete-time bimodal systems was attempted by [7]. However, this attempt was faced with a serious obstacle due to the loss of convexity for the set of reachable states of the corresponding linear system with input constraints. In [7], a case-by-case analysis produced necessary and sufficient conditions for controllability with a quite restrictive assumption on the underlying linear system. Indeed, [7] provides a characterization of controllability only for systems having at most two zeros.

In this paper, we take a completely different approach based on the recent results for the reachability of linear systems with output constraints that are presented in [5]. This new approach looks at the bimodal system at hand as a collection of linear systems with output constraints and leads to necessary and sufficient conditions for reachability without making extra assumptions on the dimension of the zero dynamics of the system.

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II. PRELIMINARIES AND NOTATION

Consider bimodal systems of the form

$$x_{t+1} = \begin{cases} A_1 x_t + B_1 u_t & \text{if } c^T x_t + d^T u_t \leqslant 0 \\ A_2 x_t + B_2 u_t & \text{if } c^T x_t + d^T u_t \geqslant 0 \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and all involved matrices are of appropriate dimensions. Throughout the paper, we assume that the right hand side of (1) is a continuous function in both x and u, equivalently

$$A_1 - A_2 = ec^T \quad \text{and} \quad B_1 - B_2 = ed^T \tag{2}$$

for some vector $e \in \mathbb{R}^n$.

We denote the set of all input sequences by $\mathcal{U}$ and the unique state trajectory of (1) for the initial state x_0 and the input u by $t \mapsto x(x_0, u, t)$.

For system (1), we define the set of reachable and null controllable states as

$$\mathcal{R} = \{\xi \mid \exists\, N \in \mathbb{N},\ u \in \mathcal{U} \text{ such that } \xi = x(0, u, N)\}$$
$$\mathcal{N} = \{\xi \mid \exists\, N \in \mathbb{N},\ u \in \mathcal{U} \text{ such that } x(\xi, u, N) = 0\}.$$

We say that the system (1) is

- *reachable* if $\mathcal{R} = \mathbb{R}^n$,
- *null-controllable* if $\mathcal{N} = \mathbb{R}^n$,
- *controllable* if $\mathcal{R} = \mathcal{N} = R^n$.

The main difficulty in the analysis of reachability of discrete-time bimodal systems stems from the fact that the reachable set is not convex in general. Loss of convexity makes the use of tools and ideas from the continuous-time counter-part problem impossible as discussed in [7, Sec. 5]. The following theorem states the main result of [7, Thm. 7.1] providing necessary and sufficient conditions for controllability.

Theorem 1 *Consider the bimodal system* (1) *with* $d = 0$*. Suppose that* $G_i(z) = c^T(zI - A_i)^{-1}B_i \not\equiv 0$ *and* $G_i(z)$ *have at most two zeros for* $i = 1, 2$*. Then, the following statements hold:*

1) *The system* (1) *is null-controllable if and only if the following implications hold:*
   a) $\lambda \in \mathbb{C} \setminus \{0\}$, $z \in \mathbb{C}^n$, $z^T A_i = \lambda z^T$, $z^T B_i = 0$ *for* $i \in \{1, 2\} \implies z = 0$.
   b) $\lambda \in \mathbb{R}_+ \setminus \{0\}$, $z \in \mathbb{R}^n$, $\begin{bmatrix} z \\ w_i \end{bmatrix}^T \begin{bmatrix} A_i - \lambda I & B_i \\ C & D \end{bmatrix} = 0$, $w_1 w_2 \leqslant 0$ *for* $i \in \{1, 2\} \implies z = 0$.
2) *The system* (1) *is controllable if and only if the following implications hold:*

a) $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$, $z^T A_i = \lambda z^T$, $z^T B_i = 0$ *for* $i \in \{1,2\} \implies z = 0$.

b) $\lambda \in \mathbb{R}_+$, $z \in \mathbb{R}^n$, $\begin{bmatrix} z \\ w_i \end{bmatrix}^T \begin{bmatrix} A_i - \lambda I & B_i \\ C & D \end{bmatrix} = 0$, $w_1 w_2 \leqslant 0$ *for* $i \in \{1,2\} \implies z = 0$.

The proof of this theorem is based on a case-by-case analysis of the possibilities for the (at most two) zeros and does not give further insight for the general case. In this paper, we take an alternative approach based on the recent results of [5] for the reachability analysis of linear systems with conical output constraints.

III. Main results

By employing the ideas and results of [5], we can state the following assertion.

Theorem 2 *Consider the bimodal system* (1). *Suppose that* $G_i(z) = d^T + c^T(zI - A_i)^{-1}B_i \not\equiv 0$ *for* $i = 1,2$. *Then, the system* (1) *is reachable if and only if the following implications hold:*

1) $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$, $z^T A_i = \lambda z^T$, $z^T B_i = 0$ *for* $i \in \{1,2\} \implies z = 0$.

2) $\lambda \in \mathbb{R}_+$, $z \in \mathbb{R}^n$, $\begin{bmatrix} z \\ w_i \end{bmatrix}^T \begin{bmatrix} A_i - \lambda I & B_i \\ C & D \end{bmatrix} = 0$, $w_1 w_2 \leqslant 0$ *for* $i \in \{1,2\} \implies z = 0$.

IV. Concluding remarks

In this paper, we provide necessary and sufficient conditions for the reachability of discrete-time piecewise bimodal systems. Unlike the existing results, we do not make any assumptions on the underlying zero dynamics. The approach we adopted is based on the recent developments for the reachability analysis of linear systems with output constraint and is rather different than those employed for the controllability analysis of piecewise affine dynamical systems in continuous-time.

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