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Nonlinear Filtering for McKean-Vlasov Type SDEs with Application to Mean Field Games

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Abstract— **This paper considers estimation theory for partially observed stochastic dynamical systems whose state equations are McKean-Vlasov type stochastic differential equations and hence contain a measure term corresponding to the distribution of the solution of the state process. Nonlinear filtering equations are provided based on the classification that either the measure term is stochastic or deterministic and that either the state or the measure term is estimated. When the measure term is deterministic the standard theory holds without any modification. In the situation where the measure term is random, the induced functions in the dynamics of the state become random and a similar recursion for the optimal filter is obtained. The joint estimation of state and the measure term is next considered. The extended state in this setup is shown to be a Polish space valued stochastic process with random functions in its state dynamics and a nonlinear filtering equation for this setup is provided. The main motivation for the development of estimation theory for McKean-Vlasov type dynamical systems arises from recent progress in mean field game theory and estimation problems in a mean field game framework are discussed. In particular, a mean field game setting with one major and many minor players is considered and nonlinear filtering formulations for the optimal estimation of both the major agent's state and the stochastic measure induced by the minor agents are provided.**

Index Terms— **Nonlinear filtering, stochastic McKean-Vlasov SDE, Random probability measures, Mean field games.**

AMS subject classifications. 60G35, 60H15, 91A25

I. INTRODUCTION

The purpose of this paper is to discuss estimation problems that appear in stochastic dynamical systems which are of McKean-Vlasov (MV) type. This class of stochastic differential equation (SDE) have the property that the drift and the diffusion coefficients of the state equation depend on the law of the state random variable; and this class of equations as well as its variants has been extensively studied in many research areas including, in particular, mathematical physics. A rigourous treatment of the existence and uniqueness results for these equations is available in the literature see e.g., [1]. In this work, however, we focus on state estimation problems for systems whose dynamics are MV type. To the best of our knowledge, and rather surprisingly, such problems have not been studied in the literature and our goal is to first define a set of estimation problems in a hierarchial manner and then to present some preliminary results in the form of nonlinear filtering equations.

MV type SDEs appear in the recently developed theory of large population non-cooperative dynamic games with mean field couplings. For such a class of games, Nash Certainty Equivalence (NCE) theory was developed in a series of papers, see [2], [3], [4] and [5] among others, by Huang, Caines and Malhamé. A closely related approach for such problems has been independently developed by Lasry and Lions in [6], [7] and [8] where the term Mean Field Game (MFG) was introduced. In summary, MFG theory considers games with a large number of stochastic dynamical agents such that each agent interacts with a mass effect (i.e., the average) of other agents via couplings in their individual cost functions and individual dynamics where each agent has a negligible influence on the overall system asymptotically in population size but the mass affect on any agent is significant. A key feature of the theory is that if the agents in a finite population system apply the infinite population equilibrium strategies, then this yields an approximate Nash equilibrium (which is also referred to as an ϵ-Nash equilibrium).

MFG theory is a rapidly growing area and has already found many potential applications such as renewable energy, communications, finance and biology. In particular, recent works consider MFG where there is a major player and many minor players (MM-MFG), see [9] and [10] for the linear quadratic Gaussian (LQG) case and [11] for the case of nonlinear state dynamics and nonlinear cost functions. In this framework, a major player has a significant influence, i.e., asymptotically non-vanishing, on any minor agents and a fundamental feature for this setup has been established [9], [11] when there is only one major player. In contrast to the situation without major agents, the mean field term becomes stochastic due to the stochastic evolution of the state of the major player. As a result, MM-MFG stochastic systems consist of a set of equations for the major player and the generic minor player which consists of a stochastic Hamilton-Jacobi-Bellman (SHJB) equation; this yields a best response control law and MV type state dynamics in which the measure term is stochastic. Such a framework has potential applications in economic and social models including power markets with large consumers and utilities together with many domestic consumers and generators [12], [13].

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In the most basic situation where there is no major player, the agents do not need to estimate the other agents' state information and the mean field term. Estimation theory starts to play a role in MFG theory when a partially observed major player is considered; in particular, for the partially observed LQG MM-MFG a Kalman filtering formulation is provided in the recent work [13]. A notable feature of MM-MFG theory is that the control actions generating ϵ-Nash equilibria depend on both the state of the major agent and the stochastic measure flow which corresponds to the mean field and which depends in turn on the Brownian noise of the major agent. Therefore the control of partially observed MM-MFG systems entails in principle the estimation of the major player's state, which has MV type dynamics, and the stochastic measure of the generic minor agent (i.e., the system's mean field).

The organization of the paper is as follows. In Section II we present estimation problems for stochastic dynamical systems with MV dynamics. Section III formulates estimation problems in the MM-MFG setting and we obtain nonlinear filters for the processes to be estimated. We conclude the paper with Section IV.

Throughout the paper we use the following notation. For a matrix A, A^T, $\mathbf{tr}(A)$ and A_{ij} denotes the transpose, the trace and the corresponding entry, respectively. ∇_x and ∇^2_{xx} denotes the gradient and Hassion operators w.r.t. the variable x and in one-dimensional domain, ∂_x and ∂^2_{xx} will be used instead. Let $\mathbb{S}$ be a metric space. Then, $\mathcal{B}(\mathbb{S})$ denotes the Borel σ-algebra, $\mathcal{P}(\mathbb{S})$ denotes the space of probability measures and $\mathcal{C}_b(\mathbb{S})$ denotes the space of bounded continuous functions on $\mathbb{S}$, respectively. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with an increasing filtration $\{\mathcal{F}_t\}$. All the filtrations defined in the paper are augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. All SDEs in the paper are of Itô type. Finally, we present some technical assumptions for the functions that appear in the dynamics or in the observations at the beginning of each section.

II. NONLINEAR FILTERING FOR MCKEAN-VLASOV TYPE DYNAMICS

Let a state process $\bar{z}_0(t,\omega)$ satisfy the following SDE

$$d\bar{z}_0(t,\omega) = \bar{f}_0\left[t, \bar{z}_0(t,\omega), \varphi_0(t, \bar{z}_0(t,\omega)), \bar{\mu}_t\right] dt + \bar{\sigma}_0\left[t, \bar{z}_0(t,\omega), \bar{\mu}_t\right] dw_0(t,\omega) \qquad (1)$$

with the initial condition $\bar{z}_0(0)$, where $\bar{z}_0 : [0,T]\times\Omega \to \mathbb{R}^n$, $\varphi_0 : [0,T]\times\mathbb{R}^n \to \mathcal{U}_0$, $\mathcal{U}_0$ is compact, $\bar{f}_0 : [0,T]\times\mathbb{R}^n\times\mathcal{U}_0\times\mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^n$, $\bar{\sigma}_0 : [0,T]\times\mathbb{R}^n\times\mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^{n\times m}$ and $w_0(t,\omega)$ is a standard Brownian motion in $\mathbb{R}^m$. Here

$$\bar{f}_0\left[t, \bar{z}_0(t,\omega), \varphi_0(t, \bar{z}_0(t,\omega)), \bar{\mu}_t\right] = \int_{\mathbb{R}^n} \bar{f}_0\left(t, \bar{z}_0(t,\omega), \varphi_0(t, \bar{z}_0(t,\omega)), x\right) \bar{\mu}_t(dx) \qquad (2)$$

for some probability distribution $\bar{\mu}_t$ on $\mathbb{R}^n$. The above type of SDEs are referred to as controlled (in the case $\varphi_0(t, \bar{z}_0(t,\omega))$ is a feedback control law) McKean-Vlasov (MV) type. Given the functions $(\bar{f}_0, \varphi_0, \bar{\sigma}_0)$, the pair $(\bar{\mu}_t, \bar{z}_0(t,\omega))$ is called to be a consistent pair if $\bar{z}_0(t,\omega)$ solves (1) and $\mathbb{P}\left(\bar{z}_0(t,\omega)\in A\right) = \int_A \bar{\mu}_t(dx)$ where $A \in \mathcal{B}(\mathbb{R}^n)$. It should be observed that the above MV dynamics can be extended to the situation where the measure term is also random. More explicitly, let $\mathcal{F}^{w_0}_t = \sigma\{w_0(s), 0\leq s\leq t\}$ and let $\hat{\mu}_t(x,\omega)$ be the conditional law of $\hat{z}_0(t,\omega)$ given $\mathcal{F}^{w_0}_t = \sigma\{w_0(s), 0\leq s\leq t\}$. Let us now denote the dynamics of the state process $\hat{z}_0(t,\omega)$ by

$$d\hat{z}_0(t,\omega) = \hat{f}_0\left[t, \hat{z}_0(t,\omega), \varphi_0(t, \hat{z}_0(t,\omega)), \hat{\mu}_t(\omega)\right] dt + \hat{\sigma}_0\left[t, \hat{z}_0(t,\omega), \hat{\mu}_t(\omega)\right] dw_0(t,\omega) \qquad (3)$$

with initial conditions $(\hat{z}_0(0), \hat{\mu}_0(\omega))$. Such dynamics are referred as stochastic coefficient McKean-Vlasov (SMV).

Notice also that assuming that the density exists, the measure terms $\bar{\mu}_t$ and $\hat{\mu}_t(x,\omega)$ satisfy Fokker-Plank-Kolmogorov (FPK) equation. More explicitly, $\hat{\mu}_t(x,\omega)$ satisfies stochastic FPK (SFPK) given as follows:

$$\frac{\partial\hat{\mu}_t(x,\omega)}{\partial t} = \Bigg(-\left\langle \nabla_x, \hat{f}_0\left[t, x, \varphi_0\left(t, \bar{z}_0(t,\omega)\right), \hat{\mu}_t(x,\omega)\right]\hat{\mu}_t(x,\omega)\right\rangle + \frac{1}{2}\mathbf{tr}\langle\nabla^2_{xx}, \hat{a}(t,x,\omega)\hat{\mu}_t(x,\omega)\rangle\Bigg) := \mathcal{L}_{\mathit{SFPK}}(t,\omega) \qquad (4)$$

in $[0,T]\times\mathbb{R}^n$ with initial value $\hat{\mu}_0(\omega)$ and $\hat{a}(t,x,\omega) := \hat{\sigma}_0[t,x,\hat{\mu}_t(\omega)]\hat{\sigma}^T_0[t,x,\hat{\mu}_t(\omega)]$. When the measure is deterministic, $\hat{\mu}_t(x,\omega)$ should be replaced with $\bar{\mu}_t$ in (4) and let $\mathcal{L}_{\mathit{FPK}}(t)$ denote the FPK equation for this case.

For these dynamics, we now formulate estimation problems in a hierarchial manner and obtain nonlinear filtering equations. In the rest of the paper, we might not display the dependence on the underlying probability space, i.e. drop the ω term in the variables of interest, to simply the notation.

Assumption ***A1****: The functions* $\bar{f}_0(\cdot), \bar{\sigma}_0(\cdot)$, $\hat{f}_0(\cdot)$ *and* $\hat{\sigma}_0(\cdot)$ *and* $h_0(\cdot)$ *are assumed to be bounded and Lipschitz continuous in their parameters [1], [16].*

State Estimation for MV SDE

Assume that the state dynamics is given by (1) and the observation dynamics is given as follows:

$$d\bar{y}_0(t,\omega,\omega') = h_0\left(t, \bar{z}_0(t,\omega)\right) dt + d\nu_0(t,\omega') \qquad (5)$$

where $h_0 : [0,T]\times\mathbb{R}^n \to \mathbb{R}^d$ is a measurable function and ν_0 is a standard Brownian motion in $\mathbb{R}^d$ and it is assumed throughout the paper that it is independent of $w_0(t,\omega)$ defined in (1) and of $\bar{z}_0(0)$. The nonlinear filtering problem is now defined as follows: Given the history of observations $\mathcal{F}^{\bar{y}_0}_t := \sigma\{\bar{y}_0(s) : s\leq t\}$, determine a recursive expression for $\mathbb{E}[\bar{z}_0(t,\omega)|\mathcal{F}^{\bar{y}_0}_t]$ and equivalently for $\bar{\pi}_0(t,\cdot) := \mathbb{P}\left(\bar{z}_0(t,\omega)\in A|\mathcal{F}^{\bar{y}_0}_t\right)$ for $A\in\mathcal{B}(\mathbb{R}^n)$. Observe that $\bar{\pi}_0(t,\cdot) : \mathcal{B}(\mathbb{R}^n)\times\Omega \to [0,1]$ is a regular conditional distribution (i.e., $\bar{\pi}_0(t,A,\omega') = \mathbb{E}_{\mathbb{P}}[1_A(\bar{z}_0(t,\omega))|\mathcal{F}^{\bar{y}_0}_t]$) and hence, satisfies i) For every $\omega'\in\Omega$, $\bar{\pi}_0(t,\cdot,\omega')$ is a probability measure on $\mathbb{R}^n$ ii) For any $A\in\mathcal{B}(\mathbb{R}^n)$, $\bar{\pi}_0(t,A,\cdot)$ is a $\mathcal{F}^{\bar{y}_0}_t$-measurable random variable and finally iii) For any $A\in\mathcal{B}(\mathbb{R}^n)$, $\bar{\pi}_0(t,A,\omega') = \mathbb{P}\left(\bar{z}_0(t,\omega)\in A|\mathcal{F}^{\bar{y}_0}_t\right)(\omega')$, $\omega'-a.s.$ The goal is to obtain a SDE for $\bar{\pi}_0(t,\cdot,\omega')$. We follow the standard steps of nonlinear filtering theory that can be found in e.g., [14], [16] and we consider the estimation of a function of $\bar{z}_0(t,\omega)$; i.e., for $\ell \in \mathcal{C}_b(\mathbb{R}^n)$, $\mathbb{E}\left[\ell\left(\bar{z}_0(t,\omega)\right)|\mathcal{F}_t^{y_0}\right]$.

Notice first that for fixed $(\bar{\mu}_t)_{0\leq t\leq T}$, $\bar{f}_0\left[t,x,u,\bar{\mu}_t\right]$, $u \in \mathcal{U}_0$, and $\bar{\sigma}_0\left[t,x,\bar{\mu}_t\right]$ become functions of (t,x,u) and (t,x), respectively, and we shall set $\bar{f}_0^*\left(t,x,u\right) := \bar{f}_0\left[t,x,u,\bar{\mu}_t\right]$ and $\bar{\sigma}_0^*\left(t,x\right) := \bar{\sigma}_0\left[t,x,\bar{\mu}_t\right]$. As a result, the estimation problem reduces to a standard nonlinear filtering problem. Following the approach presented in [16], we first define the exponential martingale that we will use frequently throughout the paper and invoke it for the change of measure argument. Let $\int_0^t \langle h_0\left(s,\bar{z}_0(s,\omega)\right), d\nu_0(s)\rangle := \sum_{j=1}^d \int_0^t h_0^j\left(s,\bar{z}_0(s,\omega)\right) d\nu_0^j(s)$ and

$$M(t)^{-1} := \mathbf{exp}\bigg(-\int_0^t \langle h_0\left(s,\bar{z}_0(s,\omega)\right), d\nu_0(s)\rangle \\ -\frac{1}{2}\int_0^t |h_0\left(s,\bar{z}_0(s,\omega)\right)|^2 ds\bigg). \tag{6}$$

Let $\mathcal{F}_t = \sigma\{w_0(s), \nu_0(s), 0 \leq s \leq t\}$ and observe that $M(t)^{-1}$ is a $\mathcal{F}_t$ martingale. Define the probability measure $\hat{\mathbb{P}}$ which is absolutely continuous with respect to $\mathbb{P}$ and the Radon-Nikodym derivative on $(\Omega, \mathcal{F}_t)$ be such that $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = M(t)^{-1}$. Hence, by Girsanov's theorem [14, Theorem 7.1.3] $\bar{y}_0(t)$ is a $\hat{\mathbb{P}}$-Brownian motion. Furthermore, by the Kallianpur-Stribel formula we have that for all $\ell \in \mathcal{C}_b(\mathbb{R}^n)$

$$\mathbb{E}_{\mathbb{P}}\left[\ell\left(\bar{z}_0(t,\omega)\right)|\mathcal{F}_t^{\bar{y}_0}\right] = \frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\bar{z}_0(t,\omega))|\mathcal{F}_t^{\bar{y}_0}\right]}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{\bar{y}_0}\right]} \tag{7}$$

Henceforth, for each problem, we first obtain a recursive an expression for the term $\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\bar{z}_0(t,\omega))|\mathcal{F}_t^{\bar{y}_0}\right]$, which is called the Duncan-Mortensen-Zakai (DMZ) equation or the unnormalized filter, and then obtain a recursion for the optimal filter (i.e., for the expectation under the measure $\mathbb{P}$).

Following the standard theory, we first note that $\bar{y}_0(t)$ is a $\hat{\mathbb{P}}$-Brownian motion and hence, by Itô's formula

$$dM(t) = M(t)h_0^T\left(t,\bar{z}_0(t,\omega)\right)d\bar{y}_0(t). \tag{8}$$

Let $T := \bar{\sigma}_0^*\left(t,x\right)\bar{\sigma}_0^{*T}\left(t,x\right)$ and set the operator on $\bar{z}_0(t,\omega)$

$$\bar{\mathcal{L}}\ell := \frac{1}{2}\sum_{j=1}^n\sum_{l=1}^n T_{jl}\partial_{jl}^2\ell + \sum_{j=1}^n \bar{f}_0^{*j}\left(t,x,u\right)\partial_j\ell. \tag{9}$$

Theorem 1: Under the assumption **A1**, the unnormalized filter for $\bar{z}_0(t,\omega)$ satisfies

$$\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\bar{z}_0(t,\omega))|\mathcal{F}_t^{\bar{y}_0}\right] = \mathbb{E}_{\hat{\mathbb{P}}}\left[M(0)\ell(\bar{z}_0(0))|\mathcal{F}_0^{\bar{y}_0}\right] \\ + \int_0^t \mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\bar{\mathcal{L}}\ell(\bar{z}_0(s,\omega))|\mathcal{F}_s^{\bar{y}_0}\right]ds + \\ \int_0^t \mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\ell\left(\bar{z}_0(s,\omega)\right)h_0^T\left(s,\bar{z}_0(s,\omega)\right)|\mathcal{F}_s^{\bar{y}_0}\right]d\bar{y}_0(s) \tag{10}$$

with initial conditional distribution $\bar{\pi}_0(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$. □

With the DMZ equation in hand, we can derive the optimal filtering equation for $\bar{\pi}_0(t)$. Let us first define the following innovation process:

$$\bar{I}_0(t) := \bar{y}_0(t) - \int_0^t \mathbb{E}_{\mathbb{P}}\left[h_0(s,\bar{z}_0(s,\omega))|\mathcal{F}_s^{\bar{y}_0}\right]ds.$$

Then it can be shown that $\bar{I}_0(t)$ is an $\mathcal{F}_t^{\bar{y}_0}$-Brownian motion under the original measure $\mathbb{P}$.

Theorem 2: Under the assumption **A1**, the optimal filter $\bar{\pi}_0(t)$ satisfies the following stochastic integral equations (SIE): For all $\ell \in \mathcal{C}_b(\mathbb{R}^n)$,

$$\mathbb{E}_{\mathbb{P}}\left[\ell\left(\bar{z}_0(t,\omega)\right)|\mathcal{F}_t^{\bar{y}_0}\right] = \\ \mathbb{E}_{\mathbb{P}}\left[\ell\left(\bar{z}_0(0)\right)|\mathcal{F}_0^{\bar{y}_0}\right] + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\bar{\mathcal{L}}\ell\left(\bar{z}_0(s,\omega)\right)|\mathcal{F}_s^{\bar{y}_0}\right]ds \\ + \int_0^t \bigg(\mathbb{E}_{\mathbb{P}}\left[\ell\left(\bar{z}_0(s,\omega)\right)h_0^T\left(s,\bar{z}_0(s,\omega)\right)|\mathcal{F}_s^{\bar{y}_0}\right] \\ - \mathbb{E}_{\mathbb{P}}\left[\ell\left(\bar{z}_0(s,\omega)\right)|\mathcal{F}_s^{\bar{y}_0}\right]\mathbb{E}_{\mathbb{P}}\left[h_0^T\left(s,\bar{z}_0\right)|\mathcal{F}_s^{\bar{y}_0}\right]\bigg)d\bar{I}_0(s) \tag{11}$$

with initial conditional distribution $\bar{\pi}_0(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$. □

State Estimation for SMV SDE

We now generalize the above result for the random measure case. In particular, the state dynamics are now given by (3) and we are interested in estimating the state process $\hat{z}(t,\omega)$. Hence, let the observation dynamics be given as follows:

$$d\hat{y}_0(t,\omega,\omega') = h_0\left(t,\hat{z}_0(t,\omega)\right)dt + d\nu_0(t,\omega') \tag{12}$$

where as before $h_0 : [0,T]\times\mathbb{R}^n \rightarrow \mathbb{R}^d$ and ν_0 is a d-dimensional standard Brownian motion and it is assumed throughout the paper that it is independent of $w_0(t,\omega)$ given in (3) and of the initial conditions $\bar{z}_0(0)$. Notice again that for a fixed stochastic measure flow $\hat{\mu}_t(\omega)_{(0\leq t\leq T)}$, $\hat{f}_0\left[t,x,u,\hat{\mu}_t\right]$ and $\hat{\sigma}_0\left[t,x,\hat{\mu}_t(\omega)\right]$ now become random functions and we write them as $\hat{f}_0^*\left(t,x,u,\omega\right) := \hat{f}_0\left[t,x,u,\hat{\mu}_t(\omega)\right]$ and $\hat{\sigma}_0^*\left(t,x,\omega\right) := \hat{\sigma}_0\left[t,x,\hat{\mu}_t(\omega)\right]$. Hence, the filtering problem reduces to the estimation of the state process whose dynamics are now driven by random functions. It is easy to show that the randomness of the function in the dynamics does not affect the analysis of nonlinear filtering, see Section III, and hence we obtain the following results. Let $\mathcal{F}_t^{\hat{y}_0} := \sigma\{\hat{y}_0(s) : s \leq t\}$, $\hat{\pi}_0(t,\cdot) := \mathbb{P}\left(\hat{z}_0(t,\omega) \in \cdot|\mathcal{F}_t^{\hat{y}_0}\right)$. Define $M(t)^{-1}$ as defined in (6) by replacing $\bar{z}_0(s,\omega)$ with $\hat{z}_0(s,\omega)$ and define the measure $\hat{\mathbb{P}}$ similarly. Let $T := \hat{\sigma}_0^*\left(t,x,\omega\right)\hat{\sigma}_0^{*T}\left(t,x,\omega\right)$ and define the following operator on $\hat{z}_0(t,\omega)$:

$$\hat{\mathcal{L}}(\omega)\ell := \frac{1}{2}\sum_{j=1}^n\sum_{l=1}^n T_{jl}\partial_{jl}^2\ell + \sum_{j=1}^n \hat{f}_0^{*j}\left(t,x,u,\omega\right)\partial_j\ell. \tag{13}$$

Theorem 3: Under the assumption **A1**, the unnormalized filter for $\hat{z}_0(t,\omega)$ satisfies

$$\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\hat{z}_0(t,\omega))|\mathcal{F}_t^{\hat{y}_0}\right] = \mathbb{E}_{\hat{\mathbb{P}}}\left[M(0)\ell(\hat{z}_0(0))|\mathcal{F}_0^{\hat{y}_0}\right] \\ + \int_0^t \mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\hat{\mathcal{L}}(\omega)\ell(\hat{z}_0(s,\omega))|\mathcal{F}_s^{\hat{y}_0}\right]ds + \\ \int_0^t \mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\ell\left(\hat{z}_0(s,\omega)\right)h_0^T\left(s,\hat{z}_0(s,\omega)\right)|\mathcal{F}_s^{\hat{y}_0}\right]d\hat{y}_0(s)$$

with initial conditional distribution $\hat{\pi}_0(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$. $\square$
Now, let $\hat{I}_0(t) := \hat{y}_0(t) - \int_0^t \mathbb{E}_{\mathbb{P}}[h_0(s,\hat{z}_0(s,\omega))|\mathcal{F}_s^{\hat{y}_0}]ds$ which is a $\mathcal{F}_t^{\hat{y}_0}$-Brownian motion under the original measure $\mathbb{P}$.

Theorem 4: Under the assumption **A1**, the optimal filter $\hat{\pi}_0(t)$ satisfies the following SIE: For all $\ell \in \mathcal{C}_b(\mathbb{R}^n)$,

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}\left[\ell\left(\hat{z}_0(t,\omega)\right)|\mathcal{F}_s^{\hat{y}_0}\right] &= \mathbb{E}_{\mathbb{P}}\left[\ell\left(\hat{z}_0(0)\right)|\mathcal{F}_0^{\hat{y}_0}\right] \\ &+ \int_0^t \mathbb{E}_{\mathbb{P}}\left[\hat{\mathcal{L}}(\omega)\ell\left(\hat{z}_0(s,\omega)\right)|\mathcal{F}_s^{\hat{y}_0}\right]ds \\ &+ \int_0^t \Bigg(\mathbb{E}_{\mathbb{P}}\left[\ell\left(\hat{z}_0(s,\omega)\right)h_0^T\left(s,\hat{z}_0\right)|\mathcal{F}_s^{\hat{y}_0}\right] \\ &- \mathbb{E}_{\mathbb{P}}\left[\ell\left(\hat{z}_0(s,\omega)\right)|\mathcal{F}_s^{\hat{y}_0}\right]\mathbb{E}_{\mathbb{P}}\left[h_0^T\left(s,\hat{z}_0\right)|\mathcal{F}_s^{\hat{y}_0}\right]\Bigg)d\hat{I}_0(s) \qquad (14)\end{aligned}$$

with initial conditional distribution $\hat{\pi}_0(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$. $\square$
In scenarios of interest below it will be required to jointly estimate the state process and the measure flow. In the rest of this section, we consider such joint estimation problems for which we first introduce some preliminary material about the metric on a space of probability measures. Let $C_n := C\left([0,T];\mathbb{R}^n\right)$ be the space of continuous functions on $[0,T]$ and let $\mathcal{F}^n$ denote the σ-algebra induced by all cylindrical sets of the form $\{x(\cdot) \in C_n : x_{t_i} \in B_i,\ t_i \in [0,T], i = 1,\cdots,l\}$ where $B_i \in \mathcal{B}(\mathbb{R}^n)$, forall i and $l \in \mathbb{N}_+$. Let $\mathcal{P}(C_n)$ denote the space of Borel probability measures μ on $(C_n,\mathcal{F}^n)$. Notice first that by following the canonical respresentaion of stochastic processes as well as using the Wasserstein metric, $D_T^{\rho}(\cdot,\cdot)$ on $\mathcal{P}(C_n)$, with appropritate metric ρ defined in $C\left([0,T];\mathbb{R}^n\right)$, it is shown in [3] that the metric space $\mathcal{P}_{\rho} := \left(\mathcal{P}\left(C_n\right), D_T^{\rho}\right)$ is Polish. However, such a representation might not be appropriate for our problem formulation. In particular, note that $h(t,\cdot)$, the observation function, is a function of $\mu_t(\omega)$ rather than the flow; $\{\mu_t(\omega), 0 \leq t \leq T\}$. Therefore, let's define the $\mathbb{R}_+ \cup \{\infty\}$-valued map $\mathcal{W}(\mu_1,\mu_2) := \inf_{\pi}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} d(x,y)d\pi(x,y)$ where $d = |\cdot|$ is the Euclidian norm and $\pi \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^n)$ with marginals μ_1 and μ_2. $\mathcal{W}(\cdot)$ is called the Wasserstein metric of order 1. Let $\mathcal{P}_n := (\mathcal{P}(\mathbb{R}^n),\mathcal{W})$ and it is known that $\mathcal{P}_n$ is a complete, separable metric space. Notice that one could use a weaker metric, such as Prokhorov metric on $\mathcal{P}(\mathbb{R}^n)$ induced by d, to metricize the space of probability measures and to obtain a Polish space. However, for the consistency throughout the paper we prefer to use $\mathcal{W}(\cdot)$. We now have two metric spaces, $(\mathbb{R}^n,d)$ and $\mathcal{P}_n$, where both of them are Polish and it is known [15] that a countable product of Polish spaces is also Polish. Based on this, let $\mathcal{C}_\cap := \mathbb{R}^n \times \mathcal{P}_n$ and let $\mathbf{z}_{o,\mu}(t,\omega)$ be a $\mathcal{C}_\cap$-valued stochastic process. Let $\xi_i(t,\cdot,\cdot) := \mathbb{P}\left(\mathbf{z}_{o,\mu}(t,\omega) \in \cdot|\mathcal{F}_t^{y_i}\right)$ where $\xi_i(\cdot) : [0,T] \times \mathcal{B}(\mathcal{C}_\cap) \times \Omega \to [0,1]$ and as before satisfies the following: i) For every $\bar{\omega} \in \Omega$, $\xi_i(t,\cdot,\bar{\omega})$ is a probability measure on $\mathcal{C}_\cap$, ii) For any $A \in \mathcal{B}(\mathcal{C}_\cap)$, $\xi_i(t,A,\cdot)$ is a $\mathcal{F}_t^{y_i}$ measurable random variable and finally, iii) For any $A \in \mathcal{B}(\mathcal{C}_\cap)$, $\xi_i(t,A,\bar{\omega}) = \mathbb{P}\left(\mathbf{z}_{o,\mu}(t,\omega) \in A|\mathcal{F}_t^{y_i}\right)(\bar{\omega}),\ \bar{\omega} - a.s.$ Observe that ξ_i is a $\mathcal{P}(\mathcal{C}_\cap)$-valued stochastic process. Finally, it is also convenient to recall the definition of conditional expectation of a measure valued random variable [17], [18]. Let $\pi(\omega)$ be a $\mathcal{P}(\mathcal{C}_\cap)$-valued random variable and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-field. An $\mathbb{R}$-valued function, F, $\mathrm{F} \in \mathcal{C}_b(\mathcal{P}(\mathcal{C}_\cap))$, is affine if there exists a real constant c and a function $\mathrm{f} \in \mathcal{C}_b(\mathcal{C}_\cap)$ such that for all $\mu \in \mathcal{P}(\mathcal{C}_\cap)$, $\mathrm{F}(\mu) = c + \mu\mathrm{f}$, where $\mu\mathrm{f} := \int \mathrm{f}(x)\mu(dx)$. Then, the conditional expectation of $\pi(\omega)$ conditioned on $\mathcal{G}$ is defined as a $\mathcal{P}(\mathcal{C}_\cap)$-valued random variable $\mathbb{E}\left[\pi(\omega)|\mathcal{G}\right]$ such that $\mathrm{F}(\mathbb{E}\left[\pi(\omega)|\mathcal{G}\right]) = \mathbb{E}\left[\mathrm{F}(\pi(\omega))|\mathcal{G}\right]$ holds for all continuous affine functions $\mathrm{F} : \mathcal{P}(\mathcal{C}_\cap) \to \mathbb{R}$.

The main motivation of the above statement is that when it is desired to jointly estimate $(\hat{z}_0(t,\omega),\hat{\mu}_t(\omega))$ we can consider this joint process as a $\mathcal{C}_\cap$-valued stochastic process where $\mathcal{C}_\cap$ is Polish and hence use the arguments of nonlinear filtering theory for stochastic processes whose state space is Polish [14, Section 8.2, 8.3].

*Assumption **A2**: The functions $g_0(\cdot)$ and $\mathbf{h}(\cdot)$ defined below have a measure term as a parameter. We assume that they are bounded and Lipschitz continuous in their parameters: e.g., for $x_1,x_2 \in \mathbb{R}^n$ and $\mu_1,\mu_2 \in \mathcal{P}(\mathbb{R}^n)$, $|g_0(t,x_1,\mu_t^1) - g_0(t,x_2,\mu_t^2)| \leq K\left(|x_1 - x_2| + \mathcal{W}(\mu_1,\mu_2)\right)$.*

State and Measure Estimation for MV SDE

In the third class of problems, we switch back to the deterministic measure case and consider the joint estimation of the state and the measure: We have the state dynamics given by (1) and the observation dynamics is given by

$$d\bar{v}_0(t) = g_0\left(t,\bar{z}_0(t,\omega),\bar{\mu}_t\right)dt + d\nu_0(t,\omega') \qquad (15)$$

where $g_0 := [0,T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^d$ and we are interested in estimating $(\bar{z}_0(t,\omega),\bar{\mu}_t)$ where $\bar{\mu}_t$ satisfies $\mathcal{L}_{\mathit{FPK}}(t)$, $\nu_0(t)$ is a standard Brownian motion in $\mathbb{R}^d$ and is independent of $w_0(t)$. Notice that since the measure term is deterministic, we have $\mathbb{P}\left(\bar{z}_0(t,\omega),\bar{\mu}_t|\mathcal{F}_t^{\bar{v}_0}\right) = \delta_{\bar{\mu}_t}\mathbb{P}\left(\bar{z}_0(t,\omega)|\mathcal{F}_t^{\bar{v}_0}\right)$. Hence, the filtering problem for this setup is no more general than the one defined via (1) and (5).

State and Measure Estimation for SMV SDE

In this problem we consider the joint estimation of the state and the measure when the measure term is itself random. Hence, we have the state dynamics given by (3) and let the observation dynamics be given by

$$d\hat{v}_0(t,\omega,\omega') = g_0\left(t,\hat{z}_0(t,\omega),\hat{\mu}_t(\omega)\right)dt + d\nu_0(t,\omega') \qquad (16)$$

Let $\hat{\mathbf{z}}_{0,\mu}(t,\omega) := (\hat{z}_0(t,\omega),\hat{\mu}_t(\omega))$ and hence, $\hat{\mathbf{z}}_{0,\mu} : [0,T] \times \Omega \to \mathcal{C}_\cap$. Recall also that we have

$$\begin{aligned}d\hat{z}_0(t,\omega) &= \hat{f}_0^*\left(t,\hat{z}_0(t,\omega),\varphi_0(t,\hat{z}_0(t,\omega)),\omega\right)dt \\ &\quad + \hat{\sigma}_0^*\left(t,\hat{z}_0(t,\omega),\omega\right)dw_0(t,\omega). \qquad (17)\end{aligned}$$

As before, our goal it to obtain a recursive expression for $\xi(t,A) := \mathbb{P}\left(\hat{\mathbf{z}}_{0,\mu}(t,\omega) \in A|\mathcal{F}_t^{\hat{v}_0}\right)$ for $A \in \mathcal{B}(\mathcal{C}_\cap)$. Note that $\hat{\mu}_t(\omega)$ satisfies (4) and hence, let $d\mathbf{w}_0(t,\omega) = (dw_0(t,\omega),0)^T$ and

$$\hat{\mathbf{F}} := \begin{bmatrix} \hat{f}_0^*\left(t,\hat{z}_0(t,\omega),\varphi_0(\cdot),\omega\right) \\ \mathcal{L}_{\mathit{SFPK}}(t,\omega) \end{bmatrix}$$

$$\hat{\mathbf{G}} := \begin{bmatrix} \hat{\sigma}_0^*\left(t,\hat{z}_0(t,\omega),\omega\right) & \mathbf{0}^T \\ \mathbf{0} & 0 \end{bmatrix}$$

where $\mathbf{0}$ is a m-dimensional 0-vector. Observe now that the estimation problem given by (16) and (17) can be written as

$$\begin{aligned} d\hat{\mathbf{z}}_{0,\mu}(t,\omega) &= \hat{\mathbf{F}}(t,\omega)dt + \hat{\mathbf{G}}(t,\omega)d\mathbf{w}_0(t,\omega) \quad (18) \\ d\hat{v}_0(t,\omega,\omega') &= g_0\left(t, \hat{\mathbf{z}}_{0,\mu}(t,\omega)\right)dt + d\nu_0(t,\omega'). \, (19) \end{aligned}$$

Therefore, the problem becomes a nonlinear filtering problem on the Polish space $\mathcal{C}_{\cap}$ with random functionals $\hat{\mathbf{F}}(t,\omega)$ and $\hat{\mathbf{G}}(t,\omega)$. In the previous sections we have observed that the standard theory holds when the state dynamics are given by random functions. Following similar steps we define $M(t)^{-1} := \mathbf{exp}\bigg(- \int_0^t \langle g_0\left(s, \hat{\mathbf{z}}_{\mathbf{0},\mu}(t,\omega)\right), d\nu_0(s)\rangle - \frac{1}{2}\int_0^t |g_0\left(s, \hat{\mathbf{z}}_{0,\mu}\right)|^2 ds\bigg)$, $\mathcal{F}_t = \sigma\{w_0(s), \nu_0(s), 0 \le s \le t\}$ and define the measure $\hat{\mathbb{P}}$ on $(\Omega, \mathcal{F}_t)$ as $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} = M(t)^{-1}$ and therefore, $\hat{v}_0(t)$ is a $\hat{\mathbb{P}}$-Brownian motion. Let $\hat{\mathbb{G}}(t,\omega) := \hat{\mathbf{G}}\hat{\mathbf{G}}^T$ and define the following operator on $\hat{\mathbf{z}}_{0,\mu}(t,\omega)$:

$$\mathcal{K}_{\hat{\mu}_t}\ell := \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\hat{\mathbb{G}}_{jl}\partial^2_{jl}\ell + \sum_{j=1}^{n}\hat{f}_0^{*j}\partial_j\ell + \frac{\partial\ell}{\partial\hat{\mu}_t}\mathcal{L}_{SFPK}(t,\omega).$$

We have the first result whose proof is similar to the proof of Theorem 9. Let $\ell \in \mathcal{C}_b(\mathcal{C}_{\cap})$.

Theorem 5: Under the assumptions **A1-A2**, the unnormalized filter for $\hat{\mathbf{z}}_{0,\mu}(t,\omega)$ satisfies

$$\begin{aligned} &\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\hat{\mathbf{z}}_{t,\mu}(0,\omega))|\mathcal{F}_t^{\hat{v}_0}\right] = \mathbb{E}_{\hat{\mathbb{P}}}\left[\ell(\hat{\mathbf{z}}_{0,\mu}(0,\omega))|\mathcal{F}_0^{\hat{v}_0}\right] \\ &\quad + \int_0^t \mathbb{E}_{\hat{\mathbb{P}}}\left[\mathcal{K}_{\hat{\mu}_s(\omega)}\ell(\hat{\mathbf{z}}_{0,\mu}(s,\omega))|\mathcal{F}_s^{\hat{v}_0}\right]ds \\ &\quad + \int_0^t \mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\ell(\hat{\mathbf{z}}_{0,\mu})g_0^T\left(s, \hat{\mathbf{z}}_{0,\mu}\right)|\mathcal{F}_s^{\hat{v}_0}\right]d\hat{v}_0(s) \qquad (20) \end{aligned}$$

with initial conditional distribution $\xi(0,\cdot) \in \mathcal{P}(\mathcal{C}_{\cap})$. □

The innovations process is defined similarly: $\hat{I}_0(t) := \hat{v}_0(t) - \int_0^t \mathbb{E}_{\mathbb{P}}\left[g_0(s, \hat{\mathbf{z}}_{0,\mu}(s,\omega))|\mathcal{F}_s^{\hat{v}_0}\right]ds$ and we obtain the following theorem whose proof is similar to the Theorem 10 below.

Theorem 6: Under the assumptions **A1-A2**, the optimal filter satisfies the following SIE: For all $\ell \in \mathcal{C}_b(\mathcal{C}_{\cap})$

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}}\left[\ell(\hat{\mathbf{z}}_{0,\mu}(t,\omega))|\mathcal{F}_t^{\hat{v}_0}\right] = \mathbb{E}_{\mathbb{P}}\left[\ell(\hat{\mathbf{z}}_{0,\mu}(0,\omega))|\mathcal{F}_0^{\hat{v}_0}\right] + \\ &\quad + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\mathcal{K}_{\mu_s(\omega)}\ell(\hat{\mathbf{z}}_{0,\mu}(s,\omega))|\mathcal{F}_s^{\hat{v}_0}\right]ds \\ &\quad + \int_0^t \bigg[\mathbb{E}_{\mathbb{P}}\left[\ell(\hat{\mathbf{z}}_{0,\mu})g_0^T\left(s, \hat{\mathbf{z}}_{0,\mu}\right)|\mathcal{F}_s^{\hat{v}_0}\right] - \\ &\quad \mathbb{E}_{\mathbb{P}}\left[\ell(\hat{\mathbf{z}}_{0,\mu}(s,\omega))|\mathcal{F}_s^{\hat{v}_0}\right]\mathbb{E}_{\mathbb{P}}\left[g_0^T\left(s, \mathbf{z}_{o,\mu}\right)|\mathcal{F}_s^{\hat{v}_0}\right]\bigg]d\hat{I}_0(s) \, (21) \end{aligned}$$

with initial conditional distribution $\xi(0,\cdot) \in \mathcal{P}(\mathcal{C}_{\cap})$. □

MV and SMV type SDEs appear in a variety of applications and here we consider MFG and apply the nonlinear filtering theory to state and measure estimation in MFG.

III. NONLINEAR FILTERING THEORY FOR MAJOR-MINOR MEAN FIELD GAMES

Assumption (A3): The functions defined in this section, i.e., $f_o(t,x,u,y)$, $\sigma_o(t,x,y)$, $f(t,x,u,y,z)$, $\sigma(t,x,y,z)$, $L_o(t,x,u,y)$ and $L(t,x,u,y,z)$, satisfy the assumptions that are sufficient for the MM-MFG results hold, namely; (i) they are continuous and bounded in all their parameters and Lipschitz continuous in (x,y,z), (ii) their first order derivatives w.r.t. x are all uniformly continuous and bounded w.r.t. all their parameters and Lipschitz continuous in (y,z), (iii) $f_o(t,x,u,y)$ and $f(t,x,u,y,z)$ are Lipschitz continuous in u, (iv) $\exists\alpha > 0$ such that $\sigma_o(t,x,y)\sigma_o^T(t,x,y) \ge \alpha I$ and $\sigma(t,x,y,z)\sigma^T(t,x,y,z) \ge \alpha I$, for all (t,x,y,z) [11]. Finally, h is assumed to be bounded and Lipschitz continuous.

Following the setup in [11], we consider a dynamic game with one major and N minor agents. Let $z_o(t) : [0,T] \times \Omega \to \mathbb{R}^n$, $z_i(t) : [0,T] \times \Omega \to \mathbb{R}^n$, $1 \le i \le N$, denote the state of the major and minor agents, respectively, $\mathcal{U}_o$ and $\mathcal{U}_i$, $1 \le i \le N$, are some compact spaces where the control actions of the agents live. The dynamics of the agents are given as follows: For $1 \le i \le N$,

$$\begin{aligned} dz_o(t) &= \frac{1}{N}\sum_{j=1}^{N} f_o(t, z_o(t), u_o(t), z_j(t))dt \\ &\quad + \frac{1}{N}\sum_{j=1}^{N}\sigma_o(t, z_o(t), z_j(t))dw_o(t) \qquad (22) \\ dz_i(t) &= \frac{1}{N}\sum_{j=1}^{N} f(t, z_o(t), u_i(t), z_i(t), z_j(t))dt \\ &\quad + \frac{1}{N}\sum_{j=1}^{N}\sigma(t, z_o(t), z_i(t), z_j(t))dw_i(t) \qquad (23) \end{aligned}$$

where $\sup_{j\in\{o,1,\cdots,N\}}\mathbb{E}|z_j(0)|^2 \le k < \infty$, where k is independent of N, $u_o(t) : [0,T] \to \mathcal{U}_o$ and $u_i(t) : [0,T] \to \mathcal{U}_i$ are the control inputs of the major and the minor agent i, respectively, $f_o : [0,T] \times \mathbb{R}^n \times \mathcal{U}_o \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_o : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $f : [0,T] \times \mathbb{R}^n \times \mathcal{U}_i \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are measurable functions and $\{(w_o(t), w_i(t))_{t\ge 0},\ 1 \le i \le N\}$, are independent standard Brownian motions in $\mathbb{R}^m$. The initial states of the agents, $\{z_o(0), z_i(0), 1 \le i \le N\}$, are independent of each other and of $\{(w_o(0), w_i(0)),\ 1 \le i \le N\}$. Let $\mathcal{F}_t = \sigma\{z_o(0), z_i(0), w_o(s), w_i(s),\ 1 \le i \le N,\ 0 \le s \le t\}$ and $\mathcal{F}_t^{w_o} = \sigma\{z_o(0), w_o(s), 0 \le s \le t\}$ be the two filtrations. The objective functions are given as follows: For $1 \le i \le N$,

$$J_o^N(u_o, u_1^N) := \mathbb{E}\int_0^T \frac{1}{N}\sum_{j=1}^{N} L_o(t, z_o(t), u_o(t), z_j(t))dt \, (24)$$

$$\begin{aligned} &J_i^N(u_i, u_{-i}) := \\ &\qquad \mathbb{E}\int_0^T \frac{1}{N}\sum_{j=1}^{N} L(t, z_o(t), u_i(t), z_i(t), z_j(t))dt \quad (25) \end{aligned}$$

where $u_1^N := \{u_1, \cdots, u_N\}$, $u_{-i} := \{u_1^N \setminus u_i\} \cup u_o$, T is the terminal time and L_o and L are $\mathbb{R}_+$-valued nonlinear cost functions. Let us now summarize the result derived in [11, Theorem 7.2]. Recall first that the major-minor agent stochastic MFG (SMFG) system is given by

$$\begin{aligned} d\tilde{z}_o(t,\omega) &= f_o[t, \tilde{z}_o, \tilde{u}_o(t,\omega,\tilde{z}_o), \mu_t(\omega)]dt \\ &\quad + \sigma_o[t, \tilde{z}_o, \mu_t(\omega)]dw_o(t,\omega), \ \ \tilde{z}_o(0) = z_o(0) \quad (26) \end{aligned}$$

$$\begin{aligned}
d\tilde{z}_m(t,\omega,\omega') &= f_m[t,\tilde{z}_m,\tilde{u}_m(t,\omega,\tilde{z}_m),\tilde{z}_o(t,\omega),\mu_t(\omega)]dt \\
&\quad +\sigma[t,\tilde{z}_m,\tilde{z}_o(t,\omega),\mu_t(\omega)]dw(t,\omega') \qquad (27)\\
\tilde{u}_o(t,\omega,x) &= \tilde{u}_o\left(t,x|\big(\mu_s(\omega)\big)_{0\le s\le T}\right) \qquad (28)\\
\tilde{u}_m(t,\omega,x) &= \tilde{u}_m\left(t,x|\big(\mu_s(\omega),\tilde{z}_o(s,\omega)\big)_{0\le s\le T}\right) \qquad (29)
\end{aligned}$$

where $\big(\mu_s(\omega)\big)_{0\le s\le T}$ is the stochastic measure flow induced by the solution of the generic minor agent stochastic MV, $\tilde{z}_m(t,\omega)$, such that $\mathbb{P}\left(\tilde{z}_m(t,\omega)\le\alpha|\mathcal{F}_t^{w_o}\right)=\int_{-\infty}^{\alpha}\mu_t(\omega,dx)$ with $\tilde{z}_m(0)$ has the measure $\mu_0(dx)=dF(x)$ where F is defined in [11, Assumption A.2.] and where $\tilde{u}_o(t,\omega,x)$ and $\tilde{u}_m(t,\omega,x)$ are the best response control processes (i.e., the unique solution of stochastic Hamilton-Jacobi-Bellman equations [11, Eqn. (5.14)-5(19)]). Equivalently, $\big(\mu_s(\omega)\big)_{0\le s\le T}$ can also be characterized by the following SFPK equation with random coefficients: Assume that the conditional density of $\tilde{z}_m(t,\omega)$ given $\mathcal{F}_t^{w_o}$ exists and denote it by $p(t,\omega,x)$, we then have

$$\begin{aligned}
&dp(t,\omega,x)= \\
&\Big(-\langle\nabla_x, f_m[t,x,\tilde{u}_m(t,\omega,\tilde{z}_m),\tilde{z}_o(t,\omega),\mu_t(\omega)]p(t,\omega,x)\rangle \\
&\quad +\frac{1}{2}\mathbf{tr}\langle\nabla_{xx}^2, a(t,\omega,x)p(t,\omega,x)\rangle\Big)dt,\ \ p(0,x)=p_0(x) \\
&:=\mathcal{L}_{FPK}^{\mu_t}(t,\omega)dt \qquad (30)
\end{aligned}$$

where $a(t,\omega,x) := \sigma[t,x,\tilde{z}_o(t,\omega),\mu_t(\omega)]\sigma^T[\cdot]$ and $\mu(t,\omega,dx)=p(t,\omega,x)dx$ a.s. $0\le t\le T$. Notice that the above dynamics are MV type SDEs where we have

$$\begin{aligned}
&f_o\left[t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),\mu_t(\omega)\right] \\
&\qquad =\int_{\mathbb{R}^n} f_o\left(t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),x\right)\mu_t(dx) \qquad (31)\\
&f_m\left[t,\tilde{z}_m,\tilde{u}_m(t,\omega,\tilde{z}_m),\tilde{z}_o,\mu_t(\omega)\right] \\
&\qquad =\int_{\mathbb{R}^n} f_m\left(t,\tilde{z}_m,\tilde{u}_m(t,\omega,\tilde{z}_m),\tilde{z}_o,x\right)\mu_t(dx). \qquad (32)
\end{aligned}$$

The SMFG best responses $\tilde{u}_o(t,\omega,x)$ and $\tilde{u}_m(t,\omega,x)$ which are obtained at the infinite population setup are next applied into a finite $N+1$ major and minor population (22)-(23). Let $\mathcal{F}_t^u := \sigma\{z_o(\tau),z_i(\tau),0\le\tau\le t,1\le i\le N\}$ and $\mathcal{U}_j$, $j\in\{o,1,\cdots,N\}$, denote the set of centralized admissible control set of an agent j where $\mathcal{U}_j=\Big\{u_j(t,\omega)$: $u_j(t,\omega)$ *is* $\mathcal{F}_t^u$ *adapted* & $\mathbb{E}_{\mathbb{P}}\int_0^T|u_j(t,\omega)|^2dt<\infty\Big\}$.

Definition 1: Given $\epsilon>0$, the admissible control laws $(u_o^o,\cdots,u_N^o)$ for $N+1$ agents generates an ϵ-Nash equilibrium with respect to the costs J_j^N, $j\in\{o,1,\cdots,N\}$, if $J_j^N(u_j^o;u_{-j}^o)-\epsilon\le\inf_{u_j\in\mathcal{U}_j}J_j^N(u_j;u_{-j}^o)\le J_j^N(u_j^o;u_{-j}^o)+\epsilon$ for any $j\in\{o,1,\cdots,N\}$.

Hence, according to [11, Theorem 7.2], the set of control laws given by $u_0^o=\tilde{u}_o$, $u_i^o=\tilde{u}_m, i=1,\cdots,N$ generates an $O(\epsilon_N+1/\sqrt{N})$-Nash equilibrium, where $\lim_{N\to\infty}\epsilon_N=0$, for the cost functions defined in (24)-(25). This result assumes that the agents are provided with the state of the major agent and also the stochastic measure (mean field) induced by the minors. In this work, we consider a partially observed major-minor MFG scenario formulate different problems based on the hierarchy that we followed in the previous section. It should be noted that in order to emphasize the MFG application, we keep $\tilde{u}_o$ and $\tilde{u}_m$ in the state dynamics below, e.g., see (34). However, the nonlinear filtering results that we obtain below hold for any admissible control laws.

Major Agent′s State Estimation in MM − MFG

In the first scenario minor agents estimate only the major agent's state. Therefore, let the observation dynamics of minor agents, $1\le i\le N$, be given as follows:

$$dy_i(t,\omega,\omega'')=h\left(t,\tilde{z}_o(t,\omega)\right)dt+d\nu_i(t,\omega'') \qquad (33)$$

where $h:[0,T]\times\mathbb{R}^n\to\mathbb{R}^d$ is a bounded measurable function and $\nu_i(t,\omega')$ is a standard Brownian motion in $\mathbb{R}^d$ which is independent of $\{w_o(t),w_i(t),i=1,\cdots,N\}$. Note that for a fixed measure flow $\mu_t(\omega)_{0\le t\le T}$, we have:

$$\begin{aligned}
&d\tilde{z}_o(t,\omega)=f_o[t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),\mu_t(\omega)]dt \\
&\qquad\qquad +\sigma_o[t,\tilde{z}_o,\mu_t(\omega)]dw_o(t) \\
&=\int_{\mathbb{R}^n} f_o\left(t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),x\right)\mu_t(dx,\omega)dt \\
&\qquad\qquad +\int_{\mathbb{R}^n}\sigma_o(t,\tilde{z}_o,x)\mu_t(dx,\omega)dw_o(t) \\
&:=f_{o,\mu_t}\left(t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),\omega\right)dt+\sigma_{o,\mu_t}(t,\tilde{z}_o,\omega)dw_o(t) \quad (34)
\end{aligned}$$

where $f_{o,\mu_t}:[0,T]\times\mathbb{R}^n\times\mathcal{U}_o\times\Omega\to\mathbb{R}^n$ and $\sigma_{o,\mu_t}:[0,T]\times\mathbb{R}^n\times\Omega\to\mathbb{R}^{n\times m}$ are now random functions. Therefore, the estimation problem is given by the SDEs:

$$\begin{aligned}
d\tilde{z}_o(t,\omega) &= f_{o,\mu_t}\left(t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),\omega\right)dt \\
&\quad +\sigma_{o,\mu_t}(t,\tilde{z}_o,\omega)dw_o(t,\omega) \qquad (35)\\
dy_i(t,\omega,\omega'') &= h\left(t,\tilde{z}_o(t,\omega)\right)dt+d\nu_i(t,\omega''). \qquad (36)
\end{aligned}$$

Let $\mathcal{F}_t^{y_i}:=\sigma\{y_i(s):s\le t\}$ and we shall now obtain a recursive expression for $\mathbb{E}[\ell(\tilde{z}_o(t,\omega))|\mathcal{F}_t^{y_i}]$, where $\ell\in\mathcal{C}_b(\mathbb{R}^n)$, and equivalently for $\pi_i(t,A,\omega''):=\mathbb{P}\left(\tilde{z}_o(t,\omega)\in A|\mathcal{F}_t^{y_i}\right)$ for $A\in\mathcal{B}(\mathbb{R}^n)$. Let $\mathcal{F}_t=\sigma\{w_o(s),\nu_i(s),0\le s\le t\}$ and we define $M(t)^{-1}$ in a same manner with (6) so that $M(t)^{-1}$ is a $\mathcal{F}_t$ martingale and let $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}=M(t)^{-1}$ so that $y_i(t)$ is a $\hat{\mathbb{P}}$-Brownian motion. Let $\mathbb{T}:=\sigma_{o,\mu_t}(t,\tilde{z}_o,\omega)\sigma_{o,\mu_t}(t,\tilde{z}_o,\omega)^T$ and define the following operator on $\tilde{z}_o(t,\omega)$

$$\mathcal{L}_{\mu_t(\omega)}\ell:=\frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{n}\mathbb{T}_{jl}\partial_{jl}^2\ell+\sum_{j=1}^{n}f_{o,\mu_t}^{j}\partial_j\ell. \qquad (37)$$

Observe that $\mathcal{L}_{\mu_t(\omega)}\ell$ is identical for all minor agents. We first derive the unnormalized filter equation for which we skip the proofs due to space limitations.

Theorem 7: Under the assumption **A3**, $\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\tilde{z}_o(t,\omega))|\mathcal{F}_t^{y_i}\right]$, $1\le i\le N$, satisfies

$$\begin{aligned}
&\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\tilde{z}_o(t,\omega))|\mathcal{F}_t^{y_i}\right]=\mathbb{E}_{\hat{\mathbb{P}}}\left[M(0)\ell(\tilde{z}_o(0,\omega))|\mathcal{F}_0^{y_i}\right] \\
&\quad +\int_0^t\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\mathcal{L}_{\mu_s(\omega)}\ell(\tilde{z}_o(s,\omega))|\mathcal{F}_s^{y_i}\right]ds \\
&\quad +\int_0^t\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\ell(\tilde{z}_o(s,\omega))h^T\left(s,\tilde{z}_o\right)|\mathcal{F}_s^{y_i}\right]dy_i(s) \quad (38)
\end{aligned}$$

with initial conditional distribution $\pi_i(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$. □
We can now derive the optimal filtering equation for $\pi_i(\cdot)$. We first define the following innovation process: $I_i(t) = y_i(t) - \int_0^t \mathbb{E}_{\mathbb{P}}\left[h(s,\tilde{z}_o(s,\omega))|\mathcal{F}_s^{y_i}\right] ds$ which can be shown to be a $\mathcal{F}_t^{y_i}$-Brownian motion under the measure $\mathbb{P}$.

Theorem 8: Under the assumption **A3**, the optimal filter satisfies the following SIE: For all $\ell \in \mathcal{C}_b(\mathbb{R}^n)$, $1 \le i \le N$,

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\ell(\tilde{z}_o(t,\omega))|\mathcal{F}_t^{y_i}\right] = \mathbb{E}_{\mathbb{P}}\left[\ell(\tilde{z}_o(0,\omega))|\mathcal{F}_0^{y_i}\right] \\
&\quad + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\mathcal{L}_{\mu_s(\omega)}\ell(\tilde{z}_o(s,\omega))|\mathcal{F}_s^{y_i}\right] ds \\
&\quad + \int_0^t \Bigg(\mathbb{E}_{\mathbb{P}}\left[\ell(\tilde{z}_o(s,\omega))h^T(s,\tilde{z}_o)|\mathcal{F}_s^{y_i}\right] \\
&\qquad -\mathbb{E}_{\mathbb{P}}\left[\ell(\tilde{z}_o(s,\omega))|\mathcal{F}_s^{y_i}\right]\mathbb{E}_{\mathbb{P}}\left[h^T(s,\tilde{z}_o)|\mathcal{F}_s^{y_i}\right]\Bigg) dI_i(s)
\end{aligned} \tag{39}$$

with initial conditional distribution $\pi_i(0,\cdot) \in \mathcal{P}(\mathbb{R}^n)$. □

State and Stochastic Measure Estimation in MM − MFG

In this setup minor agents estimate both the major agent's state and the stochastic measure flow. Therefore, let the observation dynamics of minor agent i is given by:

$$dy_i(t,\omega,\omega'') \;=\; \mathbf{h}\left(t,\tilde{z}_o(t,\omega),\mu_t(\omega)\right) dt + d\nu_i(t,\omega'') \tag{40}$$

where $\mathbf{h} : [0,T] \times \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}^d$, $\mu_t(\omega) \in \mathcal{P}(\mathbb{R}^n)$, is bounded and $\nu_i(t)$ is as defined above. Recall also that the state process dynamics can be written as:

$$\begin{aligned}
d\tilde{z}_o(t,\omega) \;:=\;& f_{o,\mu_t}\left(t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),\omega\right) dt \\
&+\sigma_{o,\mu_t}(t,\tilde{z}_o,\omega) dw_o(t,\omega)
\end{aligned} \tag{41}$$

where f_{o,μ_t} and σ_{o,μ_t} are random functions defined in (34). Hence, the problem is given by the following SDEs:

$$\begin{aligned}
d\tilde{z}_o(t,\omega) \;=\;& f_{o,\mu_t}\left(t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),\omega)\right) dt \\
&+\sigma_{o,\mu_t}(t,\tilde{z}_o,\omega) dw_o(t) \qquad (42) \\
dy_i(t,\omega,\omega'') \;=\;& \mathbf{h}\left(t,\tilde{z}_o(t,\omega),\mu_t(\omega)\right) dt + d\nu_i(t,\omega''). \qquad (43)
\end{aligned}$$

Let $\mathbf{z}_{o,\mu}(t,\omega) := (\tilde{z}_o(t,\omega),\mu_t(\omega))$, $\ell \in \mathcal{C}_b(\mathcal{C}_\cap)$ and $\xi_i(t,A,\omega'') := \mathbb{P}\left(\mathbf{z}_{o,\mu}(t,\omega) \in A|\mathcal{F}_t^{y_i}\right)$ for $A \in \mathcal{B}(\mathcal{C}_\cap)$. Recall that $\mu_t(\omega)$ satisfies the SFPK equation given in (30). Setting $d\mathbf{w}_o(t,\omega) = (dw_o(t,\omega),0)^T$ and

$$\begin{aligned}
\mathbf{F}^{\mu_t} &= \begin{bmatrix} f_{o,\mu_t}\left(t,\tilde{z}_o,\tilde{u}_o(t,\omega,\tilde{z}_o),\omega\right) \\ \mathcal{L}^{\mu_t}_{\mathit{FPK}}(t,\omega) \end{bmatrix}, \\
\mathbf{G}^{\mu_t} &= \begin{bmatrix} \sigma_{o,\mu_t}(t,\tilde{z}_o,\omega) & \mathbf{0}^T \\ \mathbf{0} & 0 \end{bmatrix},
\end{aligned} \tag{44}$$

the estimation problem given by (42)-(43) can be written as

$$\begin{aligned}
d\mathbf{z}_{o,\mu}(t,\omega) \;&=\; \mathbf{F}^{\mu_t}(t,\omega)dt + \mathbf{G}^{\mu_t}(t,\omega)d\mathbf{w}_o(t,\omega) \qquad (45) \\
dy_i(t,\omega) \;&=\; \mathbf{h}\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right) dt + d\nu_i(t) \qquad (46)
\end{aligned}$$

where $\mathbf{z}_{o,\mu} : [0,T] \times \Omega \to \mathcal{C}_\cap$ which is then a standard nonlinear filtering problem on the Polish space $\mathcal{C}_\cap$. Observe that the functionals $\mathbf{F}^{\mu_t}(t,\omega)$ and $\mathbf{G}^{\mu_t}(t,\omega)$ are random. To continue, following the steps of previous sections, let

$$\begin{aligned}
M(t)^{-1} := \mathbf{exp}\Bigg(&-\int_0^t \langle \mathbf{h}\left(s,\mathbf{z}_{o,\mu}(s,\omega)\right), d\nu_i(s)\rangle \\
&-\frac{1}{2}\int_0^t |\mathbf{h}\left(s,\mathbf{z}_{o,\mu}(s,\omega)\right)|^2 ds\Bigg)
\end{aligned} \tag{47}$$

and $\mathcal{F}_t = \sigma\{w_o(s),\nu_i(s), 0 \le s \le t\}$ so that $M(t)^{-1}$ is a $\mathcal{F}_t$-martingale. Set $\hat{\mathbb{P}}$ as before so that $y_i(t)$ is a $\hat{\mathbb{P}}$-Brownian motion and for $\ell \in \mathcal{C}_b(\mathcal{C}_\cap)$

$$\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right] = \frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]} \tag{48}$$

and $dM(t) = M(t)h^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right) dy_i(t)$. We can now present the main result. Let $\mathbb{G}(t,\omega) := \mathbf{G}^{\mu_t}\mathbf{G}^{\mu_t^T}$ and define the following operator on $\mathbf{z}_{o,\mu}(t,\omega)$

$$\Lambda_{\mu_t(\omega)}\ell := \frac{1}{2}\sum_{j,l=1}^{n} \mathbb{G}_{jl}\partial^2_{jl}\ell + \sum_{j=1}^{n} f^j_{o,\mu_t}\partial_j\ell + \frac{\partial \ell}{\partial \mu_t}\mathcal{L}^{\mu_t}_{\mathit{FPK}}. \tag{49}$$

Theorem 9: Under the assumptions **A2-A3**, for $1 \le i \le N$, $\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]$ satisfies

$$\begin{aligned}
&\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right] = \mathbb{E}_{\hat{\mathbb{P}}}\left[\ell(\mathbf{z}_{o,\mu}(0,\omega))|\mathcal{F}_0^{y_i}\right] \\
&\quad + \int_0^t \mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\Lambda_{\mu_s(\omega)}\ell(\mathbf{z}_{o,\mu}(s,\omega))|\mathcal{F}_s^{y_i}\right] ds \\
&\quad + \int_0^t \mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\ell(\mathbf{z}_{o,\mu})\mathbf{h}^T\left(s,\mathbf{z}_{o,\mu}\right)|\mathcal{F}_s^{y_i}\right] dy_i(s)
\end{aligned} \tag{50}$$

with initial conditional distribution $\xi_i(0,\cdot) \in \mathcal{P}(\mathcal{C}_\cap)$. □
The proof is an extension of the proof of Theorem 7 to Polish spaces, or more generally, extension of the nonlinear filtering theory for Polish space valued stochastic processes to the systems whose dynamics driven by random functions and we skip due to the space limitations. We now provide the optimal filtering equation for this setup. Let $\ell \in \mathcal{C}_b(\mathcal{C}_\cap)$.

Theorem 10: Under the assumptions **A2-A3**, the optimal filter $\xi_i(\cdot)$, $1 \le i \le N$, satisfies the following SIE:

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right] = \mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(0,\omega))|\mathcal{F}_0^{y_i}\right] \\
&\quad + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\Lambda_{\mu_s(\omega)}\ell(\mathbf{z}_{o,\mu}(s,\omega))|\mathcal{F}_s^{y_i}\right] ds + \\
&\int_0^t \Bigg[\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu})\mathbf{h}^T\left(s,\mathbf{z}_{o,\mu}(s,\omega)\right)|\mathcal{F}_s^{y_i}\right] - \\
&\quad \mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(s,\omega))|\mathcal{F}_s^{y_i}\right]\mathbb{E}_{\mathbb{P}}\left[\mathbf{h}^T\left(s,\mathbf{z}_{o,\mu}\right)|\mathcal{F}_s^{y_i}\right]\Bigg] dI_i(s)
\end{aligned} \tag{51}$$

with initial conditional distribution $\xi_i(0,\cdot) \in \mathcal{P}(\mathcal{C}_\cap)$. □

Proof: Notice that by taking $\ell = 1$ in (50) we obtain

$$d\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right] \overset{(i)}{=} \mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right] dy_i(t)$$

and as a result of Itô's formula

$$\begin{aligned}
&\frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_s^{y_i}\right]} = \int_0^t \frac{-1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)|\mathcal{F}_s^{y_i}\right]^2} d\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)|\mathcal{F}_s^{y_i}\right] \\
&+ \int_0^t \frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)|\mathcal{F}_s^{y_i}\right]^3} d\langle \mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)|\mathcal{F}_\cdot^{y_i}\right]\rangle_s + \frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(0)|\mathcal{F}_0^{y_i}\right]} \\
&= \int_0^t \frac{-1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)|\mathcal{F}_s^{y_i}\right]^2} d\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)|\mathcal{F}_s^{y_i}\right] + \frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(0)|\mathcal{F}_0^{y_i}\right]} \\
&\qquad + \int_0^t \frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\mathbf{h}^T\left(s,\mathbf{z}_{o,\mu}(s,\omega)\right)|\mathcal{F}_s^{y_i}\right]}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)|\mathcal{F}_s^{y_i}\right]^3}
\end{aligned}$$

$$\times\mathbb{E}_{\hat{\mathbb{P}}}\left[M(s)\mathbf{h}\left(s,\mathbf{z}_{o,\mu}(s,\omega)\right)|\mathcal{F}_s^{y_i}\right]ds. \tag{52}$$

where $\langle\mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)|\mathcal{F}_{\cdot}^{y_i}\right]\rangle_s$ denotes the quadratic variation process of $\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]$ and (52) holds due to (i) and $y_i(t)$ being a Brownian motion (see [14, Property 9, pp.57]). Let us now apply Itô's formula to (48) with $\ell \in \mathcal{C}_b(\mathcal{C}_\cap)$;

$$\begin{aligned}
&d\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right] = d\left(\frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]}\right)\\
&= \frac{d\left(\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]\right)}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]}\\
&\qquad +\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right] d\left(\frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]}\right)\\
&\qquad +d\left\langle \mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)\ell(\mathbf{z}_{o,\mu}(\cdot,\omega))|\mathcal{F}_{\cdot}^{y_i}\right], \frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)|\mathcal{F}_{\cdot}^{y_i}\right]}\right\rangle_t\\
&= \Bigg[\frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\Lambda_{\mu_t(\omega)}\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]dt}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]}\\
&\quad +\frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu}(t,\omega))\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]dy_i(t)}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]}\Bigg]\\
&\quad \Bigg[-\frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right] d\left(\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]\right)}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]^2}\\
&\qquad +\frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu})|\mathcal{F}_t^{y_i}\right]\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}\right)|\mathcal{F}_t^{y_i}\right]}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]^3}\\
&\qquad\qquad \times\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\mathbf{h}\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_s^{y_i}\right]dt\Bigg]\\
&\quad \Bigg[-\frac{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\ell(\mathbf{z}_{o,\mu})\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}\right)|\mathcal{F}_t^{y_i}\right]}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)|\mathcal{F}_t^{y_i}\right]^2}\\
&\qquad\qquad \times\mathbb{E}_{\hat{\mathbb{P}}}\left[M(t)\mathbf{h}\left(t,\mathbf{z}_{o,\mu}\right)|\mathcal{F}_t^{y_i}\right]dt\Bigg] \qquad (53)\\
&= \mathbb{E}_{\mathbb{P}}\left[\Lambda_{\mu_t(\omega)}\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]dt\\
&\qquad +\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]dy_i(t)\\
&\quad -\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]\mathbb{E}_{\mathbb{P}}\left[\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]dy_i(t)\\
&\qquad +\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]\mathbb{E}_{\mathbb{P}}\left[\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]\\
&\qquad\qquad \times\mathbb{E}_{\mathbb{P}}\left[\mathbf{h}\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]dt\\
&\qquad -\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))h^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]\\
&\qquad\qquad \times\mathbb{E}_{\mathbb{P}}\left[\mathbf{h}\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]dt \qquad (54)\\
&= \mathbb{E}_{\mathbb{P}}\left[\Lambda_{\mu_t(\omega)}\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]dt\\
&\qquad +\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]dI_i(t)\\
&\quad -\mathbb{E}_{\mathbb{P}}\left[\ell(\mathbf{z}_{o,\mu}(t,\omega))|\mathcal{F}_t^{y_i}\right]\mathbb{E}_{\mathbb{P}}\left[\mathbf{h}^T\left(t,\mathbf{z}_{o,\mu}(t,\omega)\right)|\mathcal{F}_t^{y_i}\right]dI_i(t)
\end{aligned}$$

where the first square paranthesis in (53) holds due to (50), the second one holds due to (52) and the last term is valid since the quadradic variation is given by

$$\begin{aligned}
&\left\langle \mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)\ell(\mathbf{z}_{o,\mu}(\cdot,\omega))|\mathcal{F}_{\cdot}^{y_i}\right], \frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)|\mathcal{F}_{\cdot}^{y_i}\right]}\right\rangle_t\\
&= \frac{1}{4}\Bigg(\left\langle\mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)\ell(\mathbf{z}_{o,\mu}(\cdot,\omega))|\mathcal{F}_{\cdot}^{y_i}\right] + \frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)|\mathcal{F}_{\cdot}^{y_i}\right]}\right\rangle_t\\
&\quad -\left\langle \mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)\ell(\mathbf{z}_{o,\mu}(\cdot,\omega))|\mathcal{F}_{\cdot}^{y_i}\right] - \frac{1}{\mathbb{E}_{\hat{\mathbb{P}}}\left[M(\cdot)|\mathcal{F}_{\cdot}^{y_i}\right]}\right\rangle_t\Bigg)
\end{aligned} \tag{55}$$

which can be shown to be equal to the last term in (53). The rest of the steps can be verified via (48), (50) and (52) where in (54) we substitute the innovations process $I_i(t) + \int_0^t \mathbb{E}_{\mathbb{P}}\left[\mathbf{h}(s,\mathbf{z}_{o,\mu}(s,\omega))|\mathcal{F}_s^{y_i}\right]ds$. ■

IV. CONCLUSION

In this work we have formulated estimation problems for partially observed stochastic systems whose state dynamics are driven by MV type SDEs and have provided nonliner filtering equations for several scenarios. The main motivation to study estimation theory for systems with MV type dynamics arises from MM-MFG theory and it is believed that estimation theory for MV type stochastic dynamical systems, and in particular for MFG systems, possesses interesting problems meriting investigation.

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