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Output Synchronization of Nonlinear Systems under Input Disturbances

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Abstract— **We study synchronization of nonlinear systems that satisfy an incremental passivity property. We consider the case where the control input is subject to a class of disturbances, including constant and sinusoidal disturbances with unknown phases and magnitudes and known frequencies. We design a distributed control law that recovers the synchronization of the nonlinear systems in the presence of the disturbances. Simulation results of Goodwin oscillators illustrate the effectiveness of the control law.**

I. Introduction

Synchronization of diffusively-coupled nonlinear systems is an active and rich research area [1], with applications to multi-agent systems, power systems, oscillator circuits, and physiological processes, among others. Several works in the literature study the case of static interconnections between nodes in full state models [2]–[8] or phase variables in phase coupled oscillator models [9]–[12]. Additionally, the adaptation of interconnection weights according to local synchronization errors between agents is attracting increasing attention. The authors of [13] proposed a phase-coupled oscillator model in which local interactions were reinforced between agents with similar behavior and weakened between agents with divergent behavior, leading to enhanced local synchronization. Several recent works have considered adaptation strategies based on local synchronization errors [14]–[16]. Related problems for infinite-dimensional systems have been considered in [17], [18].

Common to much of the literature is the assumption that the agents to be synchronized are homogeneous with identical dynamics, and are furthermore not subject to disturbances. However, recent work has considered synchronization and consensus in the presence of exogenous inputs. In [19], the authors addressed the problem of robust dynamic average consensus (DAC), in which the use of partial model information about a broad class of time-varying inputs enabled exact tracking of the average of the inputs through the use of the internal model principle [20] and the structure of the proportional-integral average consensus estimator formulated in [21]. The problem of DAC is highly relevant to distributed estimation and sensor fusion [22]–[25]. In [26], the authors proposed an application of the internal model principle and the robust DAC estimator in [19] to distributed Kalman filtering. In [27], the internal model principle was used in connection with passivity to achieve adaptive motion coordination. The internal model principle has also been useful in establishing necessary and sufficient conditions for output regulation [28] and synchronization [29]–[31]. Reference [32] proposed internal model control strategies in which controllers were placed on the edges of the interconnection graph to achieve output synchronization under time-varying disturbances. Recent work has also addressed robust synchronization in cyclic feedback systems [33] and in the presence of structured uncertainties [34].

In this paper, we consider synchronization of nonlinear systems that satisfy an incremental passivity property and are subject to a class of input disturbances, including constants and sinusoids with unknown phases and magnitudes and known frequencies. Reference [35] studied a similar synchronization problem without considering input disturbances. Constant and sinusoidal disturbances are common in control systems, due to biases in outputs of sensors and actuators, vibrations, etc.

Building on the robust DAC estimator developed in [19], we design a distributed control law that achieves output synchronization in the presence of disturbances by defining an internal model subsystem at each node corresponding to the disturbance inputs. Our results make several contributions differing from the existing literature on output synchronization. A fundamental achievement of our approach is that it applies to arbitrary systems that satisfy an incremental output-feedback passivity property. A similar property was employed in [36] for diffusively-coupled systems without disturbances. Using the incremental passivity property as a starting point, we establish an internal model-based controller guaranteeing asymptotic synchronization with no residual errors in the presence of a class of disturbances.

A key property of our approach is that local communication, computation and memory requirements are independent of the number of the systems in the network and the network topology, which is of interest in dense networks under processing and communication constraints. In contrast to the edge-based approach [32], which defines an internal model subsystem for each edge in the graph, our approach introduces such a subsystem only to each node, offering the advantage of a reduced number of internal states.

The rest of the paper is organized as follows. Section II reviews the output synchronization of incrementally passive systems, and provides examples using Goodwin oscillators illustrating the effect of disturbances. Our main result on output synchronization under disturbances is presented Section III. In Section IV, we illustrate the effectiveness of our control law using the example of Goodwin oscillators presented in Section II. Conclusions and future work are discussed in Section V.

Notation: The vectors 1_N and 0_N represent the N by 1 vectors with all entries 1 and 0, respectively. The space

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consisting of all N by M real matrices is given by $\mathbb{R}^{N\times M}$ while the set of real numbers is denoted by $\mathbb{R}$. Let I_N be the $N\times N$ identity matrix. The notation $\text{diag}\{k_1,\cdots,k_n\}$ denotes the n by n diagonal matrix with k_i on the diagonal. Let the transpose of a real matrix A be denoted by A^T.

II. Output synchronization without input disturbances

In this section, we briefly review the output synchronization results presented in [35], and provide an illustrative example using Goodwin oscillators.

Consider a group of N identical Single-Input-Single-Output (SISO) nonlinear systems $\mathcal{H}_i$, $i=1,\cdots,N$, given by

$$\mathcal{H}_i: \quad \dot{x}_i = f(x_i)+g(x_i)u_i, \quad x_i\in\mathbb{R}^{m\times 1} \tag{1}$$

$$y_i = h(x_i). \tag{2}$$

We assume that $\mathcal{H}_i$ satisfies the incremental output-feedback passivity (IOFP) property, i.e., given two solutions of $\mathcal{H}_i$, $x_{i_1}(t)$ and $x_{i_2}(t)$, whose input-output pairs are $(u_{i_1}(t),y_{i_1}(t))$ and $(u_{i_2}(t),y_{i_2}(t))$, there exists a positive semi-definite incremental storage function $S(\delta x(t))\in C^1$, with $S(0)=0$ such that

$$\dot{S}(\delta x(t)) \leq -\gamma(\delta y)^2+\delta y\delta u \tag{3}$$

where $\delta x = x_{i_1}-x_{i_2}$, $\delta y = y_{i_1}-y_{i_2}$ and $\delta u = u_{i_1}-u_{i_2}$ and $\gamma\in\mathbb{R}$. When $\gamma\geq 0$, $\mathcal{H}_i$ is incrementally passive (IP). When $\gamma>0$, $\mathcal{H}_i$ is incrementally output-strictly passive (IOSP). It is easy to show that for linear systems, passivity and output-strict passivity are equivalent to IP and IOSP, respectively.

Example 1: Goodwin oscillators. Consider that each $\mathcal{H}_i$, $i=1,\ldots,4$, is a Goodwin oscillator described by

$$\mathcal{H}_i: \quad \begin{aligned} \dot{x}_{i_1} &= -b_1x_{i_1}+(u_i-x_{i_4})\\ \dot{x}_{i_2} &= -b_2x_{i_2}+b_2x_{i_1}\\ \dot{x}_{i_3} &= -b_3x_{i_3}+b_3x_{i_2}\\ x_{i_4} &= -\tfrac{1}{1+x_{i_3}^p}\\ y_i &= x_{i_1}\end{aligned} \tag{4}$$

where $b_i>0$, $i=1,2,3$. In [35], the given Goodwin oscillator model (see equation (13) and Theorem 1 in [35]) was shown to be IOFP with

$$\gamma = -\frac{-1+\gamma_1\gamma_2\gamma_3\gamma_4\cos(\frac{\pi}{4})^4}{\gamma_1}, \tag{5}$$

in which γ_j is the secant gain for the dynamics of x_{i_k}, $k=1,2,3$, and γ_4 is the maximum slope of the static nonlinearity $-\frac{1}{1+z^p}$ for $z>0$. Given (4), we have $\gamma_1=\frac{1}{b}$, $\gamma_2=\frac{b_2}{b_2}=1$, and $\gamma_3=\frac{b_3}{b_3}=1$.

In this example, we choose $b_k=0.5$, $k=1,2,3$, and $p=20$. Therefore, $\gamma_1=2$, $\gamma_2=\gamma_3=1$, and $\gamma_4=5$. Therefore, the Goodwin oscillator $\mathcal{H}_i$ is IOFP with $\gamma=-0.75$, which means that $\mathcal{H}_i$ possesses a shortage of incremental passivity. □

The information flow between the outputs of the $\mathcal{H}_i$ systems is described by a bidirectional graph $\mathcal{G}$. If the bidirectional edge (i,j) exists in $\mathcal{G}$, y_i and y_j are available to $\mathcal{H}_j$ and $\mathcal{H}_i$, respectively. We denote by E the set of edges in $\mathcal{G}$. We define a weighted graph Laplacian matrix L_p of $\mathcal{G}$, whose elements are given by

$$(L_p)_{ij} = \begin{cases}\sum_{\forall j} p_{ij} & i=j\\ -p_{ij} & i\neq j,\end{cases} \tag{6}$$

where $p_{ij}=p_{ji}>0$ if $(i,j)\in E$. We set $p_{ij}=0$ if $(i,j)\notin E$. Since $\mathcal{G}$ is undirected, L_p is symmetric and satisfies $1_N^TL_p=0_N$ and $L_p1_N=0_N$. Let μ_2 be the second smallest eigenvalue of L_p. Note that L_p is positive semidefinite and thus $\mu_2\geq 0$, with the inequality strict when $\mathcal{G}$ is connected.

Theorem 2 in [35] showed that the outputs of each $\mathcal{H}_i$ are asymptotically synchronized by the following control

$$u_i = -\sum_{(i,j)\in E} p_{ij}(y_i-y_j) \quad \forall i\in\{1,\ldots,N\}, \tag{7}$$

if solutions to the closed-loop system (1), (2), and (7) exist and $\mu_2>-\gamma$. Letting $u=[u_1,\cdots,u_N]^T$ and $y=[y_1,\cdots,y_N]^T$, we obtain a compact form of (7):

$$u=-L_py. \tag{8}$$

Example 2: Synchronization of four Goodwin oscillators. We consider four Goodwin oscillators and use the control in (7) to synchronize their outputs. If we choose $u_i=0$, $\forall i$, the output of each system exhibits oscillations, as shown in Fig. 1. Because the initial conditions of the four Goodwin models are not the same, the oscillations are out of phase.

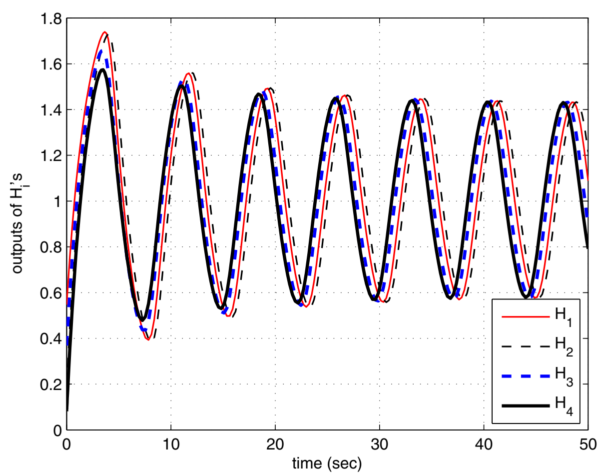

Fig. 1. Because of different initial conditions, the outputs of four Goodwin oscillators are not synchronized when $u_i=0$ in (4).

We next implement the control (7). The graph $\mathcal{G}$ is chosen to be a cycle graph and all nonzero p_{ij} in (6) are set to 1. The second smallest eigenvalue of L_p, μ_2, is 2, satisfying $\mu_2>-\gamma$. Fig. 2 shows that the outputs of these four oscillators are synchronized. □

Now suppose that the input u_i is subject to some constant input disturbance ϕ_i, $i=1,\cdots,4$. That is, $u=\phi-L_py$, where $\phi=[\phi_1,\cdots,\phi_N]^T$. The simulation result with $\phi=[0.26\ 0.8\ 0.05\ 0.55]^T$ is shown in Fig. 3, where we observe that the outputs of the four Goodwin oscillators are not synchronized due to the nonidentical disturbances ϕ_i. Due

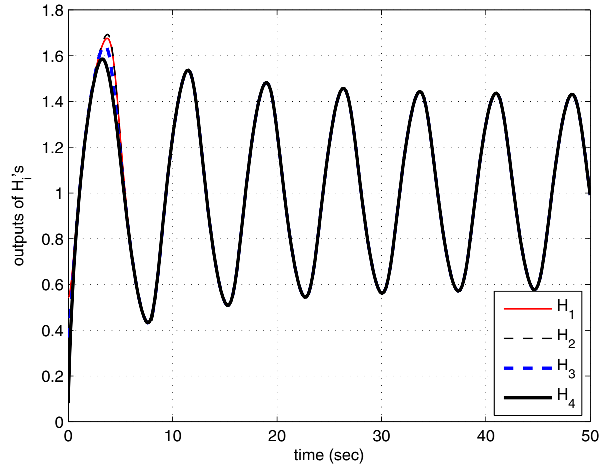

Fig. 2. Adding the proportional feedback $u = -L_p y$ leads to the synchronization of the four Goodwin oscillators.

to the constant disturbances, the outputs no longer exhibit oscillatory behavior.

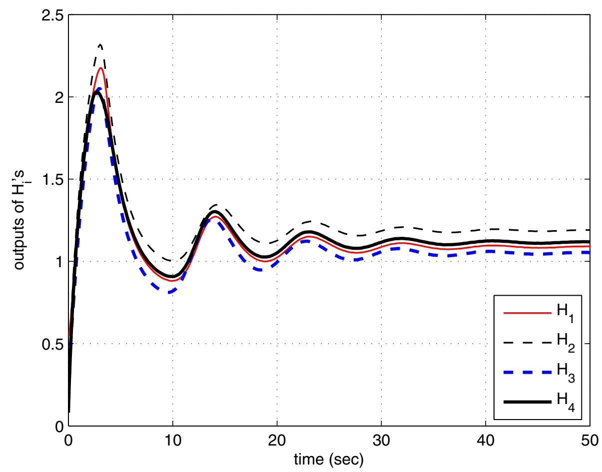

Fig. 3. The nonidentical disturbance ϕ in the feedback $u = \phi - L_p y$ destroys the synchronization of the four Goodwin oscillators.

In the next section, we present a distributed design that recovers output synchronization in the presence of a class of input disturbances, including constants and sinusoids with unknown phases and magnitudes and known frequencies.

III. Main result

We consider the scenario where the input u_i for each $\mathcal{H}_i$ is subject to a class of unknown disturbances $\phi_i(t) \in \mathbb{R}$, i.e.,

$$u_i = \bar{u}_i + \phi_i. \tag{9}$$

We assume that each disturbance ϕ_i can be characterized by

$$\dot{\xi}_i = A\xi_i, \quad \xi_i(0) \in \mathbb{R}^{n\times 1} \tag{10}$$

$$\phi_i = C\xi_i, \tag{11}$$

in which $A \in \mathbb{R}^{n\times n}$ satisfies $A = -A^T$ and the pair (A, C) is observable. Since the eigenvalues of A lie on the imaginary axis, ϕ_i can consist of both constants and sinusoids. Note that the frequencies of ϕ_i can be different. We assume that only the matrix A is available for control design.

The objective is to design the control $\bar{u}_i$ such that the outputs of $\mathcal{H}_i$, $i = 1, \cdots, N$, synchronize. Our design of $\bar{u}_i$ makes use of the model information A and embeds an internal model system G_i in the feedback loop. The information flow between the G_i systems is described by a connected and bidirectional graph $\mathcal{G}_I$. We denote by E_I the set of edges in $\mathcal{G}_I$ and consider the following design of $\bar{u}_i$:

$$\bar{u}_i = -\sum_{(i,j)\in E} p_{ij}(y_i - y_j) - \sum_{(i,j)\in E_I} n_{ij}(\eta_i - \eta_j), \tag{12}$$

where p_{ij} is defined as in (6) and $n_{ij} = n_{ji} > 0$ if $(i, j) \in E_I$. We set $n_{ij} = 0$ if $(i, j) \notin E_I$. The first term in (12) is the same as (7). For the second term, we design η_i to be the output of the internal model system G_i given by

$$G_i: \quad \dot{\zeta}_i = A\zeta_i + B_i \sum_{(i,j)\in E_I} n_{ij}(y_i - y_j) \tag{13}$$

$$\eta_i = B_i^T \zeta_i, \tag{14}$$

where (A, B_i^T) is designed to be observable and $\zeta_i(0)$, the initial condition of ζ_i, may be arbitrarily chosen.

Because $A = -A^T$, it is straightforward to show that G_i is passive from $\sum_{(i,j)\in E_I} n_{ij}(y_i - y_j)$ to η_i with a storage function $\frac{1}{2}\zeta_i^T \zeta_i$. Since G_i is a linear system, it is also incrementally passive. We will make use of the incremental passivity of G_i to prove the synchronization of the outputs y_i in the presence of ϕ_i.

***Theorem** 1: Consider the nonlinear systems $\mathcal{H}_i$ in* (1) *and* (2) *satisfying* (3) *with the input given in* (9), (12), (13) *and* (14). *Suppose that $\gamma + \mu_2 > 0$. If the solutions are bounded, then the outputs y_i synchronize asymptotically:*

$$\lim_{t\to\infty} \left| y_i(t) - \frac{1}{N} 1_N^T y(t) \right| = 0, \quad \forall i \in \{1, \ldots, N\}. \tag{15}$$

□

If $\gamma > 0$, that is, $\mathcal{H}_i$ possesses an excess of incremental passivity, Theorem 1 allows μ_2 to be zero, which means that the graph $\mathcal{G}$ can be disconnected. If $\gamma \le 0$, $\gamma + \mu_2 > 0$ leads to $\mu_2 > -\gamma > 0$, which means that $\mathcal{G}$ must be connected.

Let $\bar{u}_i = [\bar{u}_1, \cdots, \bar{u}_N]^T$ and $\eta = [\eta_1, \cdots, \eta_N]^T$ and define a weighted graph Laplacian L_I for $\mathcal{G}_I$ as

$$(L_I)_{ij} = \begin{cases} \sum_{\forall j} n_{ij} & i = j \\ -n_{ij} & i \ne j. \end{cases} \tag{16}$$

Then the control in (12) can be rewritten as

$$\bar{u} = -L_p y - L_I \eta. \tag{17}$$

The diagram in Fig. 4 shows the closed-loop system given by (1), (2), (9), (12), (13) and (14).

We next employ the incremental passivity property of both $\mathcal{H}_i$ and G_i and the symmetry of L_I to prove Theorem 1.

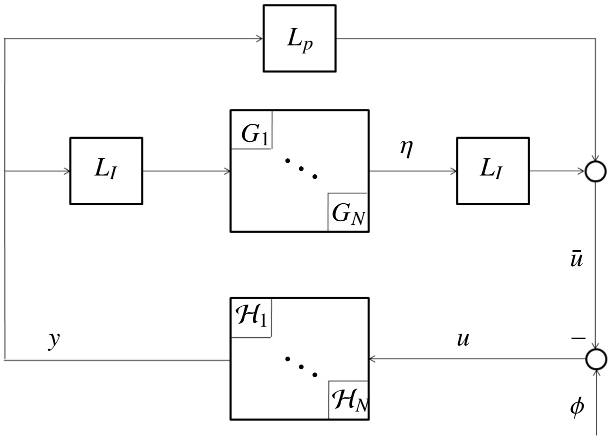

Fig. 4. The block diagram of the closed-loop system given by (1), (2), (9), (12), (13) and (14).

Proof: Define the orthogonal projection matrix $\Pi \in \mathbb{R}^{N\times N}$ by:

$$\Pi = I_N - \frac{1}{N} 1_N 1_N^T. \tag{18}$$

We first consider the storage function

$$V = \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N} S(x_i - x_j), \tag{19}$$

whose time derivative along (1) and (2) is given by

$$\dot V \le \frac{1}{2N}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[-\gamma(y_i-y_j)^2 + (y_i-y_j)(u_i-u_j)\right] \tag{20}$$

$$= -\gamma y^T \Pi y + y^T \Pi u. \tag{21}$$

The equality in (21) follows because

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left[(y_i-y_j)(u_i-u_j)\right] \tag{22}$$

$$= \sum_{i=1}^{N}\left[(y_i 1_N - y)^T(u_i 1_N - u)\right] \tag{23}$$

$$= \sum_{i=1}^{N}\left[N y_i u_i + u^T y - u_i 1_N^T y - y_i 1_N^T u\right] \tag{24}$$

$$= 2N y^T u - u^T 1_N 1_N^T y - y^T 1_N 1_N^T u = 2N y^T \Pi u. \tag{25}$$

We substitute (9) into (21) and obtain

$$\dot V \le -\gamma y^T \Pi y + y^T \Pi(\phi + \bar u). \tag{26}$$

Noting (17), we further get

$$\begin{aligned}\dot V &\le -\gamma y^T \Pi y + y^T \Pi(\phi - L_I\eta - L_p y)\\ &= -y^T(\gamma\Pi + L_p)y + y^T(\Pi\phi - L_I\eta).\end{aligned} \tag{27}$$

We introduce auxiliary systems

$$\dot z_i = A z_i, \quad z_i(0) \in \mathbb{R}^{n\times 1}, \quad i = 1,\cdots,N, \tag{28}$$

$$\lambda_i = B_i^T z_i, \tag{29}$$

where the initial conditions of z_i, $z_i(0)$, will be chosen later, and define $\lambda = [\lambda_1,\cdots,\lambda_N]^T$. We also let

$$\delta_i := \zeta_i - z_i, \tag{30}$$

with $\delta = [\delta_1,\cdots,\delta_N]^T$.

We now employ the incremental passivity of G_i and consider the following storage function:

$$W = \frac{1}{2}\sum_{i=1}^{N}\delta_i^T\delta_i. \tag{31}$$

Using (13), (14), (28) and (29), we obtain

$$\dot W = \sum_{i=1}^{N}\delta_i^T B_i \sum_{(i,j)\in E} n_{ij}(y_i - y_j) \tag{32}$$

$$= (\eta - \lambda)^T L_I y. \tag{33}$$

The sum $Z = V + W$ yields

$$\begin{aligned}\dot Z &= \dot V + \dot W\\ &\le -y^T(\gamma\Pi + L_p)y + y^T(\Pi\phi - L_I\eta) + (\eta-\lambda)^T L_I y\\ &= -y^T(\gamma\Pi + L_p)y + y^T(\Pi\phi - L_I\lambda).\end{aligned} \tag{34}$$

Because the control (12) is independent of the auxiliary systems (28)-(29), $z_i(0)$ may be arbitrarily chosen. We claim that by appropriately choosing $z_i(0)$, $i = 1,\cdots,N$, we guarantee

$$\Pi\phi = L_I\lambda. \tag{35}$$

To see this, we consider the following systems:

$$\dot{\hat\xi}_i = A\hat\xi_i, \quad \hat\xi_i(0) \in \mathbb{R}^{n\times 1}, \quad i = 1,\cdots,N \tag{36}$$

$$\hat\phi_i = C\hat\xi_i. \tag{37}$$

We define a $(N-1)\times N$ matrix Q that satisfies $Q1_N = 0$, $QQ^T = I_{N-1}$ and $Q^TQ = \Pi$. We let

$$\Gamma = Q^T(QL_IQ^T)^{-1}Q \tag{38}$$

and denote by Γ_{ij} the element at the ith row and jth column of Γ. The inverse of QL_IQ^T exists because 1_N spans the null spaces of L_I and Q. Note that Γ is the Moore-Penrose pseudoinverse of L_I.

We first show that choosing $\hat\xi_i(0) = \sum_{j=1}^{N}\Gamma_{ij}\xi_j(0)$, $\forall i$, guarantees

$$\Pi\phi = L_I\hat\phi \tag{39}$$

where $\hat\phi = [\hat\phi_1,\cdots,\hat\phi_N]^T$. Note that $\hat\xi_i(0) = \sum_{j=1}^{N}\Gamma_{ij}\xi_j(0)$ results in $\hat\phi = \Gamma\phi$. Using (38), we obtain

$$QL_I\hat\phi = Q\phi. \tag{40}$$

Pre-multiplying (40) by Q^T and noting $Q^TQL_I = \Pi L_I = L_I$ verify (39).

We next show that by selecting $z_i(0)$ in (28) appropriately, we ensure $\lambda = \hat\phi$. In particular, we choose $z_i(0) = O_{B_i}^{-1}O_C\hat\xi_i(0)$, where O_{B_i} is the observability matrix of (28)-(29) and O_C is the observability matrix of (36)-(37). Since $z_i(0) = O_{B_i}^{-1}O_C\hat\xi_i(0)$, $z_i(t) = O_{B_i}^{-1}O_C\hat\xi_i(t)$, which means $O_{B_i}z_i(t) = O_C\hat\xi_i(t)$. Noting that the first row of O_{B_i} and O_C is B_i^T and

C, respectively, we have $\lambda_i = B_i^T z_i = C\hat{\xi}_i = \phi_i$, $\forall i$. Then the claim (35) follows from (39) and $\lambda = \hat{\phi}$.

Having proved that (35) can be achieved by appropriately selecting $z_i(0)$ in (28), we obtain from (34) and (35) that

$$\dot{Z} \le -y^T(\gamma\Pi + L_p)y. \tag{41}$$

Because $y^T L_p y = (Qy)^T Q L_p Q^T (Qy) \ge \mu_2 (Qy)^T Qy$, we rewrite (41) as

$$\dot{Z} \le -(\gamma + \mu_2) y^T Q^T Q y \le 0. \tag{42}$$

By integrating both sides of (42), we see that Qy is in $\mathcal{L}_2$. Furthermore, the boundedness of solutions implies that $\dot{x}_i$ and thus $\dot{y}_i$ are bounded for all i. An application of Barbalat's Lemma [37] implies that $Qy \to 0$ as $t \to \infty$. Thus, $\Pi y \to 0$ as $t \to \infty$, which, together with (18), implies (15). ■

We note from (13) that the differences between the outputs of the ith node and its neighboring nodes are first aggregated and then passed as an input to an internal model subsystem G_i. This node-based approach is different from the edge-based approach [32], where the difference between the outputs of the ith node and each of its neighboring nodes is directly passed to an internal model subsystem. In a distributed implementation, the ith node must maintain one internal model subsystem for each of its neighboring nodes. It is possible for the ith node to take over the control for the edge (i, j) and communicate the control signal to the adjacent node j. However, such an implementation is no longer distributed and requires additional communication.

For our node-based approach, each node maintains only one internal model subsystem in total and the dimension of ζ_i is independent of the number of the nodes in the network and the number of neighbors of the ith nodes. This is advantageous in dense networks under processing and communication constraints. A comparison between the performance of the node-based and edge-based approaches is currently being pursued by the authors.

IV. Motivating example revisited

We now implement our control law, given in (9) and (12), to recover the synchronization of the outputs of the four oscillators. The graph $\mathcal{G}_I$ is chosen to be the same as $\mathcal{G}$ in Example 2. All nonzero n_{ij} in (16) are set to 1. The initial conditions and the disturbance ϕ remains the same as in Example 2. Fig. 5 shows that the outputs of the oscillators are asymptotically synchronized. Note that Theorem 1 only guarantees the synchronization of the outputs y_i and may not recover the nominal oscillations of y_i shown in Fig. 2. In fact, as manifested in the proof of Theorem 1 (cf. (27) and (41)), the internal model based control $L_I\eta$ in (17) only compensates for the effects due to $\Pi\phi$, the difference between ϕ and $\frac{1}{N}1_N 1_N^T\phi$. Therefore, if $1_N^T\phi \neq 0$, the remaining disturbance $\frac{1}{N}1_N 1_N^T\phi$ still enters the system. However, it does not affect the synchronization.

We next present two examples where the oscillations of y_i can also be recovered.

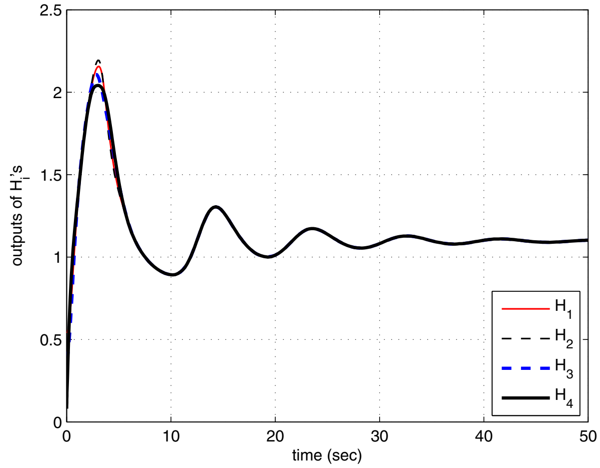

Fig. 5. The outputs of the four Goodwin oscillators are synchronized with the control (9) and (12).

1) $1_N^T\phi = 0$: As discussed above, if $1_N^T\phi = 0$, all the disturbances are compensated for by our control. We choose $\phi = [-0.155\ 0.385\ -0.365\ 0.135]^T$ such that $1_N^T\phi = 0$. The simulation results in Fig. 6 illustrate that the outputs of the four Goodwin oscillators exhibit synchronized oscillations shown in Example 2.

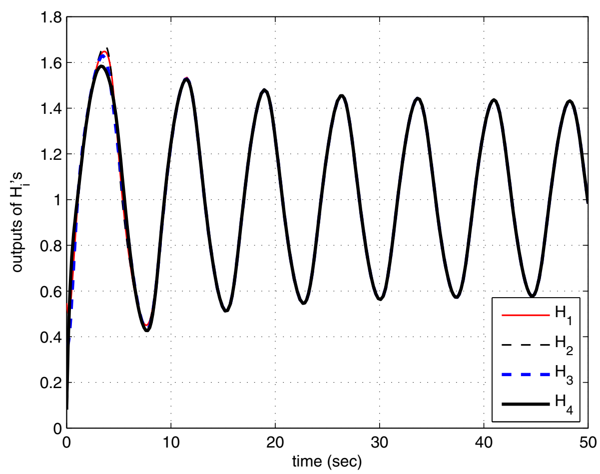

Fig. 6. The outputs of the four Goodwin oscillators when $1_N^T\phi = 0$. The control (9) and (12) recovers the synchronization of the outputs, which exhibit synchronized oscillations as in Example 2.

2) Synchronize with a reference: In this example, we suppose that for some $\mathcal{H}_i$, say, $i = 1$, no disturbance enters $\mathcal{H}_1$, that is, $\phi_1 = 0$. Then $\mathcal{H}_1$ can be considered as a reference (a leader) and it can choose to implement $u_1 = -L_p^1 y$, where L_p^1 is the first row of L_p. The other oscillators have the same disturbance inputs as in Example 2 and implement (9) and (12). With the modification, the simulation results in Fig. 7 show the recovery of both the oscillation and the synchronization of the outputs.

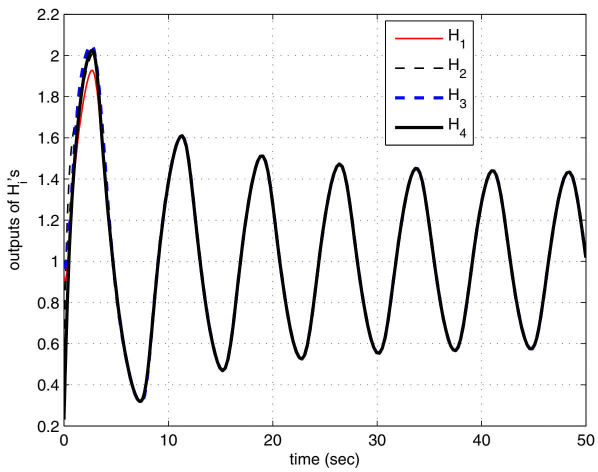

Fig. 7. The outputs of the four Goodwin oscillators when $\phi_1 = 0$, and $\mathcal{H}_1$ employs only the proportional feedback (7) while the other $\mathcal{H}_i$ $(i = 2,3,4)$ implement (9) and (12). Under this modification, the outputs also exhibit synchronized oscillations as in Example 2.

V. Conclusions

We have studied synchronization of nonlinear systems that are incrementally passive, and designed a distributed control law that recovers synchronization in the presences of disturbances of a certain class using the internal model principle. Our controller has the advantage of requiring a reduced number of additional states relative to other approaches, and furthermore does not require knowledge of the initial conditions of the disturbances. We have illustrated our results with several examples using Goodwin oscillators. In future work, we will establish the connection of the proposed control law to the dynamic average consensus estimator studied in [19], demonstrate the use of adaptive updates of the coupling graph to reduce time to synchronize, and address additional classes of disturbances and controller designs.

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