

# Consensus Between Nonlinearly Coupled Agents with Heterogeneous Input Delays: Analytic Criteria

Anton V. Proskurnikov

**Abstract**—The paper addresses the problem of consensus robustness against small heterogeneous input delays in the agents. The agents are assumed to have first-order dynamics and may be nonlinearly coupled; the couplings maps may be uncertain, assumed only to satisfy a sector (slope) restriction and a symmetry condition. The topology of the network may be switching and uncertain, however, it is supposed to be undirected and uniformly connected (which is known to be almost necessary for consensus). Under the mentioned assumptions, we obtain explicit analytical conditions for the consensus robustness.

## I. INTRODUCTION.

Recent decades witnessed the explosive growth of interest to various aspects of regular cooperative behavior in large-scale groups of simple agents, which interact only locally and do not have (or do not use) any information about the system in whole. A lot of complex systems arising in engineering and natural sciences may be treated in this multi-agent framework, large biological populations such as flocks of birds and smart power grids being typical examples. The problem of reaching *consensus* (agreement, synchrony) between the agents is now considered to be one of the most important case studies in the area since the phenomena of synchronism emerging from local interaction among the agents lies in the heart of many processes in natural and technical systems. Recent monographs [1]–[3] (see also references therein) give an excellent survey of recent results on multi-agent consensus and their applications.

Despite the enormous overall progress in understanding the consensus phenomena, some important problems still remain unexplored even for the simplest possible agents with first-order integrator dynamics. One of such problems is robustness of consensus protocols against uncertain delays in the transmitted or measured data and actuators which are inevitable in practice and may lead to deterioration of the consensus and other instability effects. The effect of delays on cooperative behavior has been studied, however, only for a few types of delayed consensus algorithms.

The simplest case quite satisfactorily studied in literature [4]–[7] is where each agent is aware of the instantaneous value of own state, and the delays (often called *communication delays*) affect only data transmitted by neighbors (or the neighbors' influence). Consensus appears to be robust

against arbitrarily large non-stationary communication delays provided that they remain bounded, which can be proved by retracing standard arguments for undelayed averaging consensus algorithms [8], [9]. The cornerstone of the proof is the shrinking property of the convex hull of agents' states, which set-valued counterpart of Lyapunov function should be replaced by the convex hull of all states observed during a sufficiently long time. Some other Lyapunov techniques which yield explicit formulas for the consensus value were proposed in [10].

In presence of self-delays which may be caused e.g. by delayed self-actuation or retarded influence of the neighbors resulting from relative measurements the mentioned contraction-based techniques do not work; moreover, for large enough delays the solutions of closed-loop systems are exponentially unbounded [11]; so the real concern dealing with self-delayed consensus protocols is to find the critical delay margin below which synchronization is established [11]–[13]. In the case of linear time-invariant networks tight estimates for this margin may be obtained from standard frequency-domain analysis [5], [12]–[15]. For time-variant topology only sufficient estimates of the delay threshold are known that employ quite complicated systems of high-dimensional LMI [16], [17]. Analogous methods are applicable also to second-order agents [18], [19]. Unlike the first-order agents, for such agents consensus may be *induced* by the delay [20] in the sense that network with undelayed communication does not reach consensus.

A common restriction of the results just mentioned is that they are applicable only for linearly coupled networks. In the same time, nonlinearly coupled consensus algorithms naturally arise in many applications such as synchronization of oscillators networks, or rendezvous with range- and rate-restricted communication [1], [9], [21]. The necessity to consider such protocols motivated rapid development of the nonlinear consensus theory, see [1], [22]–[25] and references therein. However, except for the already mentioned case of communication delay and first-order agents [4]–[7], most results in this area deal with *undelayed couplings* only, whereas nonlinear protocols with delayed data remain almost unexplored. Some progress in this direction was achieved in the previous works by the author [26]–[30], addressing an important class of nonlinear consensus algorithms with delayed couplings (incorporating self-delays) and switching topology. Assumptions about the delays were inspired by the situation where the self-delay is based on relative measurements response from neighbors (e.g. depending on signal propagation only). An analytic estimate for the critical delay

The work was supported by the European Research Council (ERCStG-307207) and RFBR, grants 12-01-00808, 13-08-01014 and 14-08-01015.

A.V. Proskurnikov is with the Research Institute of Technology and Management, University of Groningen, Nijenborgh 4, 9747 AG Groningen, the Netherlands and also with St. Petersburg State University and Institute for Problems of Mechanical Engineering RAS, St.Petersburg, Russia.

margin was proposed, based on absolute stability approach [26], [27] and a sort of small-gain techniques [28]–[30]. In the present paper, we consider another type of delays, namely input (actuator) delays in the agents. This model can not be reduced to those considered in [26]–[30] except for the special situation where the input delays are homogeneous, which is usually a doubtful condition. A typical example of a multi-agent system where each agent has in fact its own actuator delay is microscopic traffic flow models with delayed driver reaction, see [31], [32] and references therein for historical issues. The actual delay depends on a number of factors and is individual for each driver.

To simplify matters, we confine ourselves to the case where the delays are time-invariant and discrete. Furthermore, we deal only with first-order agents. An extension to more general class of agents may be obtained by using the Popov method from absolute stability theory, some results in this direction were reported in [33].

## II. PRELIMINARIES AND THE PROBLEM SETUP.

We first recall some concepts from the graph theory. A *graph* is a pair  $G = (V, E)$  constituted by the finite set of *nodes*  $V$  and the set of *arcs*  $E \subset V \times V$ . The graph is said to be *undirected* if  $(v, w) \in E \Leftrightarrow (w, v) \in E \forall v, w \in V$ ; a sequence of its nodes  $v_1, v_2, \dots, v_k$  with  $(v_i, v_{i+1}) \in E \forall i$  is called the *path* between  $v_1$  and  $v_k$ ; the graph is said to be *connected* if a path exists between any two nodes.

We consider a team of agents indexed 1 through  $N$

$$\dot{x}_j(t) = u_j(t - \tau_j), \quad j = 1, 2, \dots, N, \quad (1)$$

that are coupled by applying the following protocol

$$u_j(t) = \sum_{k=1}^N a_{jk}(t) \varphi_{jk}(t, y_{jk}(t)), \quad y_{jk} := x_k - x_j. \quad (2)$$

Here  $t \geq 0$ ,  $x_j(t) \in \mathbb{R}^n$  stands for the state of the  $j$ -th agent, the maps  $\varphi_{jk}(t, x)$  are called *couplings*,  $\tau_j \geq 0$  is the input delay. The coupling gains  $a_{jk}(t) \geq 0$  define the instantaneous interaction graph: the  $k$ -th node influences the  $j$ -th one at time  $t \geq 0$  if and only if  $a_{jk}(t) > 0$ . When dealing with discontinuous at  $t = 0$  solutions of (1), we assume them right-continuous at  $t = 0$  for the definiteness. We assume the initial functions  $x_j(t)$ ,  $t < 0$  to be bounded.

The objective is to disclose conditions under which the system comes to consensus in the following sense.

*Definition 1:* The protocol (2) *establishes consensus* if  $|x_k(t) - x_j(t)| \xrightarrow{t \rightarrow \infty} 0$  for any initial data and  $j, k$ .

In general, consensus does not imply the existence of the limits  $\lim_{t \rightarrow +\infty} x_j(t)$ ; it is a non-trivial problem to calculate the consensus point or trajectory.

## III. ASSUMPTIONS AND MAIN RESULT.

Our starting assumption concerns the network topology.

*Assumption 1:* The matrix  $A(t) = (a_{jk}(t))$  is symmetric and Lebesgue measurable. There exist  $\varepsilon > 0$  and  $T_0 > 0$  such that the graph  $(V_N, \mathcal{E}_t)$  with the set of nodes  $V_N = \{1, \dots, N\}$  and that of arcs  $\mathcal{E}_t = \{(j, k) : \int_t^{t+T_0} a_{jk}(s) ds > \varepsilon\}$  is connected for all  $t \geq 0$ .

The first claim implies that the interaction graph is undirected. The second property is often referred to as the *uniform connectivity* of the network and is commonly adopted in the literature along with its analogs for directed graphs [4], [6], [9]. It prohibits disintegration of the network into separated clusters and is acknowledged as nearly necessary for consensus. For the constant weight matrix  $A(t) \equiv A = A^T$ , Assumption 1 means that the graph  $G = (V_N, E)$ ,  $E := \{(j, k) : a_{jk} > 0\}$  is connected.

The following *symmetry assumption* imposes a relation-ship that is similar in spirit to the Newton's Third Law.

*Assumption 2:* For any  $t \geq 0$ ,  $x \in \mathbb{R}^n$ ,  $j, k \in V_N$ ,  $j \neq k$ , one has  $\varphi_{jk}(t, x) = -\varphi_{kj}(t, -x)$ .

If the input delays are homogeneous ( $\tau_1 = \dots = \tau_N$ ), under Assumption 2 consensus implies the average consensus since summing up (2) over  $j = 1, \dots, N$  yields  $\sum_j u_j(t) \equiv 0$  and thus  $\dot{s}(t) = 0$  for  $s(t) := N^{-1} \sum_j x_j(t)$ . So  $s(t) \equiv s(0)$  and  $x_k(t) - x_j(t) \rightarrow 0 \forall j, k \implies x_j(t) \rightarrow s(0) \forall j$ .

We are concerned with the situation where the couplings  $\varphi_{jk}$  are not completely known. The available knowledge about them basically comes to a certain *quadratic inequality*. Specifically for a known constant  $\gamma > 0$ , they belong  $\varphi_{jk} \in \mathfrak{S}(\gamma)$  to the set  $\mathfrak{S}(\gamma)$  of continuous maps  $\varphi : [0; +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\varphi(t, 0) \equiv 0$  and

$$\gamma x^T \varphi(t, x) \geq |\varphi(t, x)|^2 \quad \forall t \geq 0, x \neq 0; \quad (3)$$

$$\inf_{t \geq 0, |x| > \delta} |\varphi(t, x)| > 0 \quad \forall \delta > 0. \quad (4)$$

In the scalar case ( $n = 1$ ), (3) and (4) express the conventional sector condition: the graph of the function  $\varphi(t, \cdot)$  lies between the lines  $y = \gamma x$  and  $y = 0$  (intersecting the latter line at the origin only). So for  $n \geq 2$ , the inclusion  $\varphi_{jk} \in \mathfrak{S}(\gamma)$  may be viewed as a multi-variable analog of the sector condition.

The coupling weights  $a_{jk}$  are Lebesgue measurable and uncertain, we assume only that a priori bounds  $\bar{d}_j$  for the node weighted degrees are known

$$d_j(t) := \sum_{k=1}^N a_{jk}(t) \leq \bar{d}_j \quad \forall t \geq 0. \quad (5)$$

Consensus criterion should be given in terms of bounds  $\bar{d}_j$  and the "sector slope"  $\gamma$ , but not the couplings, weights, and delays themselves. Such a criterion in fact ensures the *robust* consensus in the sense that consensus holds for all uncertainties satisfying the above requirements.

The following is the main result of the paper which gives a sufficient condition for consensus.

*Theorem 1:* Suppose that Assumptions 1 and 2 hold, inequalities (13) are valid and  $\varphi_{jk} \in \mathfrak{S}(\gamma)$  for some  $\gamma > 0$ . Then the protocol (2) establishes consensus if

$$\tau_j \bar{d}_j < (2\gamma)^{-1} \quad \forall j. \quad (6)$$

The proof of Theorem 1 will be given in Section V.

## IV. DISCUSSION AND APPLICATIONS.

Below we discuss the relations between Theorem 1 and previously known results, and also give some of its applications to microscopic traffic flow models.

### A. Comparison with Existing Results.

The inequality (14) is in touch with the previous literature [12], [13], [15] where analytic conditions for consensus under fixed delays were obtained. Unlike this paper, it deals with identity couplings  $\varphi_{jk}(x) = x$  (which belong to  $\mathfrak{S}(\gamma)$  with  $\gamma := 1$  in our setting) and fixed topology  $a_{jk}(t) \equiv a_{jk}$ .

As was claimed in [13, Section IV, Remark 4], for constant topology and identity couplings (14) guarantees consensus even if the interaction graph is directed (the connectivity is to be replaced with existence of an oriented spanning tree). Thus our result may be considered as an extension of a result from [13] to the class of nonlinear consensus protocol, dealing however only with undirected topology and symmetric couplings.

For constant undirected topology, trivial couplings  $\varphi_{jk}(x) = x$  and homogeneous delays  $\tau_j = \tau$  tight estimates of the maximal tolerable delay were obtained in [11].

In this case, the estimate takes the form  $\tau_j \leq \pi/(2\lambda_{\max}(L)) \forall j, k$ , where  $\lambda_{\max}(L)$  is the maximal eigenvalue of the Laplacian matrix  $L$  which is defined as

$$L := \begin{bmatrix} \sum_{j=1}^N a_{1j} & -a_{12} & \dots & -a_{1N} \\ -a_{21} & \sum_{j=1}^N a_{2j} & \dots & -a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{N1} & -a_{N2} & \dots & \sum_{j=1}^N a_{Nj} \end{bmatrix}.$$

The right-hand side of this inequality exceeds that of (14) since  $\pi/2 > 1$  and  $\lambda_{\max} < 2\bar{d}_j$  by the Gershgorin theorem [3]. So in the case of fixed interaction topology and trivial couplings, (14) is only sufficient but not necessary for the consensus. In [5], [15] a sufficient condition for heterogeneous delays was obtained that is a sort of Nyquist criterion. It should be also emphasized that the techniques from [5], [13], [15] are not applicable to switching networks and nonlinear couplings, which are among the main concerns of this paper.

If the delays are homogeneous:  $\tau_j = \tau$ , and conditions of Theorem 1 hold, the sufficiency of (14) for consensus is close to the result of [29, Theorem 1] (since the delay may be formally "removed" from the input signal to the couplings, and the network remains symmetric in the sense of [29, Assumption 3]) with the only difference that  $\bar{d}_j$  should be replaced with  $\sum_{k=1}^N \bar{a}_{jk}$ , where  $\bar{a}_{jk}$  are known a priori bounds for the gains  $a_{jk}(t)$ . Unlike Theorem 1, the result from [29] remains valid for non-stationary delay  $\tau(t)$ . In the case of *fixed* topology and homogeneous constant delays the result of Theorem 1 may be significantly improved using the IQC method [26].

### B. Traffic Flow Models with Driver Reaction Delays

An important example of the networked system with delays is *microscopic traffic flow model* treating the flow as a result of interaction between individual drivers, which make decisions independently based on the observed positions and velocities of nearby vehicles. Microscopic traffic flow models are closely related to models of self-propelled particle ensembles [34] and commonly adopted as simple but instructive tools for traffic analysis. Since the pioneering

work [35], the *delay* in drivers reaction has been recognized as a factor essentially affecting the overall flow dynamics, see e.g. [31], [32] and references therein for a historical survey.

A simple model of such kind [31], [34], [35] deals with  $N$  vehicles, indexed 1 through  $N$ , following along a common circular single lane road. Each vehicle tries to equalize its velocity with that of its predecessor:

$$\dot{v}_j(t) = K(v_{j\oplus 1}(t - \tau) - v_j(t - \tau)).$$

Here  $v_j(t)$  is the velocity of the  $j$ -th vehicle,  $\tau$  is the delay in the driver's action,  $\oplus$  is the summation modulus  $N$ , and  $K$  stands for the driver's "sensitivity" to alterations of the relative velocity of the vehicle in front of him.

A key issue addressed via this model is that of stability of the constant-velocity equilibria  $v_1 \equiv \dots \equiv v_N \equiv \text{const}$ .

The respective results of [31], [32] deal with more general interaction topology, assuming that a driver may watch not only its predecessor but also other vehicles. The delays in driver reactions may be distributed, which reflects some properties of the human memory. However, there are several restrictive assumptions: the delays are homogeneous, the topology is fixed and the acceleration is a linear function of the relative velocities of nearby vehicles. Now we extend the results of [31], [32] on the more realistic case where each driver reacts with his/her individual delay, the observation topology alters over time (a driver loses or acquires sight of the companions depending on the relief and weather conditions) and the couplings may be nonlinear. However, unlike the mentioned papers we assume the network to be symmetric and the delays to be discrete. Some analogous results for distributed and time-variant, however, homogeneous delays in driver reactions may be found in [30].

Specifically, we assume that the driver of the  $j$ -th vehicle adjusts the velocity  $v_j$  based on the relative velocities of  $p \leq N - 1$  preceding and  $p$  following vehicles:

$$\dot{v}_j(t) = u_j(t - \tau_j), u_j(t) = K \sum_{m=-p}^p a^m(t) \varphi^m(v_{j\oplus m}(t) - v_j(t)). \quad (7)$$

Here the functions  $a^0(t) = 0$  and  $a^m(t) \geq 0$  define the topology: how "far" the driver can see, and couplings  $\varphi^m$  describe how the driver reacts on the change of relative velocity of the  $m$ -th successor and predecessor. This "order-based" determinism leads to the assumption that the response to the  $m$ -th predecessor and the  $m$ -th follower are equally sharp:  $a^m = a^{-m}$  and  $\varphi^m(v) = -\varphi^{-m}(-v) \forall m$ . The driver reaction delays  $\tau_j \geq 0$  are assumed to be constant.

Applying Theorem 1 gives rise to the following.

**Theorem 2:** Let  $a^m(t) = a^{-m}(t) \geq 0$ ,  $\varphi^m(v) = -\varphi^{-m}(-v)$  and  $\varphi^m \in \mathfrak{S}(\gamma)$  for any  $m = -p \dots, p$ . Suppose that  $T, \varepsilon > 0$  exist such that  $\int_t^{t+T} a^1(t) dt > \varepsilon$  for any  $t \geq 0$ . Let  $\bar{d} := \sup_{t \geq 0} \sum_{m=-p}^p a^m(t)$  and suppose that  $\tau_j < (2\bar{d}\gamma)^{-1}$ . Then the system (7) achieves the "velocity consensus", i.e.  $v_j(t) - v_k(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all  $j$ .

Indeed, it is easy to verify that all assumptions of Theorem 1 with  $a_{j,j\oplus m} := a^m$ ,  $\varphi_{j,j\oplus m} := \varphi^m$  and other  $a_{jk}$  equal to zero (and thus  $\bar{d}_j = \bar{d}$ ) are valid.

V. PROOF OF THEOREM 1

From now on, the assumptions of Theorem 1 are supposed to hold. Given a solution  $x_j(t), u_j(t)$  of the closed loop system (1),(2), let  $\phi_{jk}(t) := \varphi_{jk}(t, x_k(t) - x_j(t)) \in \mathbb{R}^n$ ,  $\eta_{jk}(t) := a_{jk}(t)|\phi_{jk}(t)|^2$ . From (2) it is easily seen that

$$u_j(t) = \sum_{k=1}^N a_{jk}(t)\phi_{jk}(t) \quad (8)$$

$$|u_j(t)|^2 = \left| \sum_{k=1}^N a_{jk}^{1/2}(t)a_{jk}^{1/2}(t)\phi_{jk}(t) \right|^2 \leq \overbrace{\sum_k a_{jk}(t)}^{\leq \bar{d}_j} \sum_k \eta_{jk}(t). \quad (9)$$

We denote the norm in  $L_2[0;T]$  with  $\|\cdot\|_T$ , i.e.  $\|f\|_T^2 = \int_0^T |f(s)|^2 ds$ .

Combining formulas (8),(9) together with (3) yields in the following quadratic constraint for the solutions.

*Lemma 1:* Let  $\theta_j > \tau_j$  be such numbers that  $\alpha_j := \gamma^{-1} - 2\theta_j \bar{d}_j > 0$  for any  $j$ . Then any solution of (1),(2) satisfies

$$2 \sum_{j=1}^N \left[ x_j(t)^T u_j(t) + 2\theta_j |u_j(t)|^2 + \alpha_j \sum_{k=1}^N \eta_{jk}(t) \right] \leq 0 \forall t \geq 0. \quad (10)$$

**Proof.** The inequality (3) entails that  $\phi_{jk}(t)^T (x_k(t) - x_j(t)) - \gamma^{-1} |\phi_{jk}(t)|^2 \geq 0$ . Multiplying by  $a_{jk}(t) \geq 0$  and summing up over all  $j, k$ , taking into account equalities (8),  $\alpha_j + 2\theta_j \bar{d}_j = \gamma^{-1}$  and  $\phi_{jk} = -\phi_{kj}$ , implies that

$$\sum_{j=1}^N \left[ -2x_j(t)^T u_j(t) - 2\theta_j \bar{d}_j \sum_{k=1}^N \eta_{jk}(t) - \alpha_j \sum_{k=1}^N \eta_{jk}(t) \right] \geq 0,$$

from where (10) follows accordingly to (9).  $\square$

The key part of the proof is contained in the following.

*Lemma 2:* Any solution  $(x_j(t))_{j=1}^N$  of (1),(2) is bounded, can be extended on  $[0, \infty)$ , and is such that

$$\int_0^\infty |\dot{x}_j(t)|^2 dt < \infty, \int_0^\infty \eta_{jk} dt < \infty, \forall j, k. \quad (11)$$

**Proof.** Since  $u_j(t) = \dot{x}_j(t + \tau_j)$ , one has  $u_j(t)^T x_j(t) = u_j(t)^T x_j(t + \tau_j) - u_j(t)^T [x_j(t + \tau) - x_j(t)]$ . Therefore,

$$2 \int_0^T u_j(t)^T x_j(t) dt = |x_j(t + \tau_j)|^2 \Big|_0^T - 2 \underbrace{\int_0^T u_j(t)^T \int_t^{t+\tau_j} \dot{x}_j(s) ds dt}_{=: \rho_j(T)}.$$

Applying the Cauchy-Schwartz inequality, one easily obtains

$$\begin{aligned} |\rho_j(T)|^2 &\leq \|u_j\|_T^2 \int_0^T \left| \int_t^{t+\tau_j} \dot{x}_j(s) ds \right|^2 dt \leq \\ &\leq \tau_j \|u_j\|_T^2 \int_0^T \int_{t-\tau_j}^t |u_j(s)|^2 ds dt \leq \\ &\leq \tau_j \|u_j\|_T^2 \left( \int_{-\tau_j}^0 (s + \tau_j) |u_j(s)|^2 ds + \tau_j \|u_j\|_T^2 \right). \end{aligned}$$

Therefore,  $2|\rho_j(T)| \leq 2\tau_j \|u_j\|_T^2 + C_j^0$  and  $2 \int_0^T u_j(t)^T x_j(t) dt \geq |x_j(T + \tau_j)|^2 - 2\tau_j \|u_j\|_T^2 - C_j$ , where  $C_j^0, C_j = C_j^0 + |x_j(\tau_j)|^2$  depend only on initial data. By substituting this into (10), one immediately obtains that the following function is globally bounded (when  $T \geq 0$ ):

$$\sum_{j=1}^N \left[ |x_j(T + \tau_j)|^2 + 2(\theta_j - \tau_j) \|u_j\|_T^2 + \alpha_j \sum_{k=1}^N \eta_{jk} \right] \leq \sum_{j=1}^N C_j,$$

which proves Lemma 2 since  $\theta_j > \tau_j$  and  $\alpha_j > 0$ .

**Proof of Theorem 1.** Suppose to the contrary that consensus does not hold. Then there exist a solution of (1),(2) a number  $\delta > 0$ , and a sequence  $t_m \uparrow +\infty$  such that  $\max_{j,k} |x_j(t_m) - x_k(t_m)| > 3\delta N$ . Without any loss of generality, it can be assumed that  $t_{m+1} - t_m > T_0$ , where  $T_0$  is taken from Assumption 1. Then the sets  $\Delta_m = [t_m; t_m + T_0]$  are disjoint. Since the graph  $(V_N, \mathcal{E}_{t_m})$  is connected for any  $m$ , an arc  $(j_m, k_m) \in \mathcal{E}_{t_m}$  exists such that  $|x_{j_m}(t_m) - x_{k_m}(t_m)| > 3\delta$ . Thanks to (11)  $\int_0^\infty |\dot{x}_j(t)|^2 dt < \infty \forall j$ ; hence  $|x_j(t'') - x_j(t')| \rightarrow 0$  as  $t'', t' \rightarrow +\infty$  and  $|t'' - t'| \leq T_0$ . Hence  $|x_{j_m}(t) - x_{j_m}(t_m)| \leq \delta$  and  $|x_{k_m}(t) - x_{k_m}(t_m)| \leq \delta$  for  $t \in \Delta_m$  and  $m$  sufficiently large, therefore  $|x_{j_m}(t) - x_{k_m}(t)| \geq \delta$ . Accordingly to (4), there exists  $\nu > 0$  such that  $|\phi_{j_m k_m}(t) - \phi_{jk}(t, x_{j_m}(t) - x_{k_m}(t))| \geq \nu \forall t \in \Delta_m$  and thus  $\sum_{j,k} \int_{\Delta_m} \eta_{jk}(t) dt \geq \varepsilon \nu^2$  by the definition of  $\mathcal{E}_{t_m}$ . Thus we arrive at the contradiction with (11), which completes the proof.  $\square$

VI. EXTENSION: MIXED INPUT AND MEASUREMENT DELAYS

In this section we discuss the case when heterogeneous input delays coexist with delays in relative measurements  $y_{jk}(t) = x_k(t) - x_j(t)$  that were investigated in [27]–[30].

Consider a team of agents (1) coupled via a protocol

$$u_j(t) = \sum_{k=1}^N a_{jk}(t) \varphi_{jk}(t, y_{jk}(t - \tau_{jk}(t))). \quad (12)$$

Here  $\tau_{jk}(t) \geq 0$  are unknown time-varying delays, and besides Assumptions 1,2 we assume that  $\tau_{jk}(t) = \tau_{kj}(t)$ . Suppose the upper bounds for gains and delays to be available:

$$a_{jk}(t) \leq \bar{a}_{jk}, \tau_{jk}(t) \leq \bar{\tau}_{jk}, d_j(t) \leq \bar{d}_j \forall t \geq 0, \quad (13)$$

where  $d_j(t) := \sum_{k=1}^N a_{jk}(t)$ .

The following result, incorporating Theorem 1 and [29, Theorem 5], gives sufficient condition for consensus in the closed-loop networked system (1),(12).

*Theorem 3:* Let  $\tau_{jk} = \tau_{kj}$ , Assumptions 1, 2 and (13) hold and  $\varphi_{jk} \in \mathfrak{S}(\gamma)$  for some  $\gamma > 0$ . The protocol (12) establishes consensus if for any  $j$  the following inequality holds:

$$\frac{1}{2\gamma} - \max_j (\bar{d}_j \tau_j^0) - \max_j \left( \bar{d}_j \sum_{k=1}^N \bar{a}_{jk} \bar{\tau}_{jk}^2 \right)^{1/2} > 0. \quad (14)$$

**Sketch of the proof.** Analysis of the proof of [29, Lemma 8] reveals that any solution of the system (1),(2) satisfies the following inequality:

$$\int_0^T \sum_{j=1}^N \left[ 2x_j(t)^T u_j(t) + \beta \sum_{k=1}^N \eta_{jk}(t) \right] dt \leq 0 \forall T \geq 0.$$

Here  $\eta_{jk} := a_{jk}(t)|\phi_{jk}(t, y_{jk}(t - \tau_{jk}(t)))|$  and  $\beta := \frac{1}{\gamma} - \max_j \left( \bar{d}_j \sum_{k=1}^N \bar{a}_{jk} \bar{\tau}_{jk}^2 \right)^{1/2}$ . Due to (9), the latter inequality implies an "integral" counterpart of the quadratic constraint (10). Namely, if  $\theta_i > \bar{\tau}_i^0$  ( $i = 1, \dots, N$ ) are such that  $\alpha_i := \beta - \theta_i \bar{d}_i = \gamma^{-1} - \theta_i \bar{d}_i - 2 \max_j \left( \bar{d}_j \sum_{k=1}^N \bar{a}_{jk} \bar{\tau}_{jk}^2 \right)^{1/2} > 0$ , then

$$\int_0^T \sum_{j=1}^N \left[ 2x_j(t)^T u_j(t) + \theta_j |u_j(t)|^2 + \alpha_j \sum_{k=1}^N \eta_{jk}(t) \right] dt \leq 0. \quad (15)$$

Thus, we are able to retrace the proof of Lemma 2, arriving at the inequalities (11).

## VII. CONCLUSION

We address the problem of consensus among nonlinearly coupled first-order agents with heterogeneous input delays. The network topology is undirected and satisfies the assumption of uniform connectivity. The only information about the couplings come to a symmetry condition similar in flavor to the Newton Third Law and a sector condition with known slopes. A new criterion for robust consensus is obtained in terms of a priori delay bounds and sector slopes for nonlinear couplings. Its extensions on the leader-following formation control, reference-tracking consensus, and agents with more general dynamics is subjects of ongoing research.

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