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Consensus in discrete-time nonlinearly coupled networks

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Abstract— **We consider consensus algorithms for multi-agent networks with discrete-time linear identical MIMO agents. The agents may be of arbitrary order, the interaction topology may be time-varying and the couplings may be nonlinear and uncertain, however assumed to satisfy a slope restriction or, more generally, quadratic constraint. Using the discrete-time version of the KYP Lemma (referred to as the Kalman-Szegö Lemma), we derive a criterion which provide consensus in such a network for any uncertain couplings from the mentioned class. This criterion is close in spirit to the celebrated Tsypkin criterion for discrete time Lurie system.**

I. INTRODUCTION.

Recently problems of multi-agent synchronization or *consensus* have attracted significant interest of the research community. These problems are focused on synchronism between the subsystems (called *agents* or nodes of a complex system) achieved by means of local interactions. Such synchronism, or *consensus*, lies in the heart of cooperative dynamics exhibited by many complex systems arising in physics, biology, economics etc. The review of recent results in the area and those applications, as well as historical discussions may be found in recent reviews and monographs [20], [22], [31], [32], [40] and extensive references therein.

The ideas of simplest first order consensus algorithms take their origin in agreement procedures in expert communities coming from theory of management and applied statistics and distributed algorithms in computer science. The state of a first order agent may be treated as opinion on some quantity of interest; to make the team agree on this quantity, those algorithms propose each agent to average its individual opinion with ones from the team-mates so that the dispersion of opinions decreases. Considering the opinions as points in a vector space, each of them moves into the relative interior of the convex hull spanned by itself and the opinions of neighboring agents. The convex hull of agents states is shrinking during such a procedure and, using its diameter as a Lyapunov function, this hull may be proven to collapse into a singleton [3], [17], [21]. Alternative techniques for proving consensus employ the theory of stochastic matrices [4], [20], [22] and theory of contracting mappings [5].

Despite many protocols of second and higher order directly extend their first order prototypes [20], [31], [32], these algorithms are usually no longer contracting, which makes investigation of their convergence a non trivial problem. A number of protocols with linear couplings may be examined using the Laplacian matrix decomposition techniques [6], [14], [31], [41], [43] or by direct reduction to the first order case [18], [33], [39]. However, many applications involve nonlinearly coupled networks dynamics of which cannot be analyzed using mentioned linear techniques. The examples include, but not are limited to, oscillator networks where the couplings are periodic functions [12], [35], multi-agent coordination with range-restricted communication [15], [17], [34]. In the real-world conditions linear protocols may become nonlinear because of analog-digital conversions, quantization effects and nonlinear distortions in measurements. The mentioned challenges stimulated the rapid development of nonlinear consensus theory.

But for the aforementioned results on first order agents, most existing results on convergence on consensus among nonlinearly coupled agents address the case of continuous time networks and are based on specially constructed Lyapunov functions. Most accomplished are consensus criteria for passive agents [2] where the sum of individual storage functions may be taken as an "energy-like" Lyapunov function. Some results for agents with second order dynamics coupled by $\tanh$-like nonlinear maps have been obtained in [31]. However, for arbitrary non-passive nonlinearly coupled agents no general method for establishing synchronization seems to be known. Some progress in this direction was made in recent author's papers [24], [25], [27] where networks of general LTI agents with switching interaction topology and nonlinear couplings were considered. The nonlinear couplings may be uncertain and only assumed to satisfy some natural symmetry condition and be restricted by some known quadratic cone (or sector with known slopes in the scalar case). Using the the Kalman-Yakubovich-Popov (KYP) lemma, a method for constructing quadratic Lyapunov functions for such networks was proposed in the mentioned papers that allows to obtain effective frequency-domain consensus criteria extending a number of results in the area. Those criteria are similar in flavor to the well-known circle criterion (and its multi-variable analogues) in absolute stability theory.

In the present papers, we extend the results obtained in [24], [25], [27] to the class of discrete-time consensus protocols with nonlinear couplings. Whereas the passivity-based arguments and other Lyapunov techniques elaborated for continuous-time agents in [31], [32] are not directly applicable to discrete-time networks, the methods coming from absolute stability theory and based on general quadratic Lyapunov functions, as will be shown below, work for discrete time agents as well. Unlike [24], [25], [27], we

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consider the case of weighted interaction graphs (to each coupling map a time-varying and possibly uncertain gain corresponds). The criteria obtained in the paper work for networked systems with uncertain couplings, that are assumed only to satisfy *quadratic constraints* and an anti-symmetry assumption resembling the Newton Third Law. The consensus condition involves only coefficients of the correspondent quadratic constraint and some graph-theoretic measures of the network topology, but not couplings and the interaction graph themselves, providing thus *robustness* of the consensus against the uncertainties in question.

The paper is organized as follows. Section II contains the problem set up and main assumptions. Section III presents the main results. Section IV illustrates application of the main results to some special classes of agents.

II. Preliminaries and the problem setup.

Throughout the paper $\underline{N}$ stands for the set $\{1, 2, \dots, N\}$. We put $\mathbb{R}_+ := [0; +\infty)$ and $\mathbb{C}_+ := \{z \in \mathbb{C} : Re\, z \geq 0\}$. For a matrix $A \in \mathbb{C}^{m\times n}$ we denote with $A^* \in \mathbb{C}^{n\times m}$ its complex-conjugate transpose, for $a \in \mathbb{C}$ we have $a^* = \bar{a}$. The components of a vector $u \in \mathbb{R}^m$ are numbered with upper indices, thus $u = (u^1, \dots, u^m)^T$. Given several column vectors $u_1, \dots, u_N$, their column union is denoted with $col(u_1, \dots, u_N)$. The symbol $diag(A_1, \dots, A_N)$ denotes the block-diagonal matrix composed of square matrices A_j.

Given a square matrix A, we denote its spectrum with $\sigma(A) \subset \mathbb{C}$. We say that the matrix is *stable* if its spectrum lies in the open unit disc: $|\lambda| < 1$ for any $\lambda \in \sigma(A)$.

Any non-negative $(N \times N)$-matrix $G = (g_{jk})$ may be considered as the adjacency matrix of a weighted graph $\hat{G} := [\underline{N}, E, G]$ whose nodes are indexed 1 through N and arcs correspond to positive items in G, that is $E = \{(j, k) : g_{jk} > 0\}$. Throughout the paper we bound ourselves with undirected weighted graphs without self-loops which means that $G = G^T$ and $g_{jj} = 0$. As usual, a *path* between the nodes $v, v' \in \underline{N}$ in the graph $\hat{G}$ is a sequence $v_1 = v, v_2, \dots, v_{k-1}, v_k = v'$ such that $(v_1, v_2), (v_2, v_3), \dots, (v_{k-1}, v_k) \in E$. The graph $\hat{G}$ is *connected* if a path between any two nodes exists.

We say the number $\Delta_j[G] := \sum_{k=1}^N g_{jk}$ to be the *degree* of the j-th node. The properties of the graph $\hat{G}$ are closely related to the structure of its *Laplacian*

$$L[G] := diag(\Delta_1[G], \dots, \Delta_N[G]) - G. \qquad (1)$$

The second term in the ascending sequence of eigenvalues $\lambda_1(G) = 0 \leq \lambda_2(G) \leq \dots \leq \lambda_N(G)$ of the symmetric matrix $L[G] \geq 0$ is called *the algebraic connectivity* of the graph $\hat{G}$ and; from the Courant-Fischer-Weyl theorem the following extremal property may be derived [7]

$$\lambda_2(G) = N \min_{z\in\Upsilon} \frac{\sum_{j,k=1}^N g_{jk}(z_k - z_j)^2}{\sum_{j,k=1}^N (z_k - z_j)^2} \qquad (2)$$

where $\Upsilon := \left\{z \in \mathbb{R}^N : z_k \neq z_j \text{ for some } j, k\right\}$. The graph $\hat{G}$ is connected if and only if $\lambda_2(G) > 0$ [20], [22], [31].

A. Problem set up

Throughout the paper we deal with a team of identical agents, indexed 1 through $N \geq 2$ and governed by a common MIMO state-space model

$$x_j(t+1) = Ax_j(t)+Bu_j(t),\ y_j(t) = Cx_j(t)+Du_j(t), \quad (3)$$

Here $t = 0, 1, 2, \dots$, $x_j(t) \in \mathbb{R}^d$, $u_j(t) \in \mathbb{R}^m$, $y_j(t) \in \mathbb{R}^l$ stand for the state, control, and output of the j-th agent, respectively. The model (3) is assumed to be controllable and observable. The control inputs $u_j(t)$ are affected by interactions (via communication or otherwise) between the agents. Specifically, we examine the following protocols

$$u_j(t) = \sum_{k=1}^N g_{jk}(t)\varphi_{jk}(y_k(t) - y_j(t)). \qquad (4)$$

The *coupling maps* $\varphi_{jk} : \mathbb{R}^l \to \mathbb{R}^m$ determine the "law of interaction" in each pair of agents, and the *coupling gains* $g_{jk}(t) \geq 0$ describe the "intensiveness" of such an interaction and also the topology of the network ($g_{jk}(t) > 0$ means that the k-th agent influences the j-th one at time t).

The aim of the paper is to disclose conditions under which such a protocol establishes consensus among the agents in the following sense.

Definition 1: The protocol (4) establishes consensus if

$$\mathcal{W}(t) := \sum_{j,k=1}^N |x_j(t) - x_k(t)|^2 \underset{t\to\infty}{\longrightarrow} 0 \quad \forall (x_j(0))_{j=1}^N. \quad (5)$$

If additionally $\mathcal{W}(t) \leq C\alpha^t\mathcal{W}(0)$ for some constants $C > 0, \alpha \in (0; 1)$ (independent from initial data), we say that it establishes the *exponential consensus*.

Often consensus is understood in some weaker sense, e.g. as output synchronization: $y_j(t) - y_k(t) \underset{t\to\infty}{\longrightarrow} 0$ for any initial data. Under non-restrictive assumptions the output synchronization implies full consensus.

Lemma 1: Suppose that functions $g_{jk}(t)$ are bounded, $\varphi_{jk}(\cdot)$ are continuous and $y_j(t) - y_k(t) \to 0$ as $t \to \infty$ for any j, k. Then $x_j(t) - x_k(t) \to 0$ for any j, k.

The proof retraces that of [24, Remark 2] and relies on the controllability of the system (3).

B. Main Assumptions

The assumptions about protocol (4) come to the symmetry of the network, connectivity of the interaction topology and cone restrictions on the couplings.

We start with the assumption about the interaction graph.

Assumption 1: For any $t = 0, 1, \dots$ we have $G(t) \in \mathbb{G}$, where $\mathbb{G}$ is some known compact set of non-negative matrices, that are assumed to be symmetric ($G = G^T\ \forall G \in \mathbb{G}$) with corresponding graphs $\hat{G}$ connected. For such a class of matrices, we put by definition

$$\lambda_{2*} := \min_{G\in\mathbb{G}} \lambda_2(G) > 0,\ \Delta_* := \max_{G\in\mathbb{G}} \Delta_j[G]. \qquad (6)$$

Typical example of the class $\mathbb{G}$ is the class of all non-negative matrices G with $g_{jk} \in \{0\} \cup [g'_{jk}; g''_{jk}]$ (where $0 < g'_{jk} = g'_{kj} < g''_{jk} = g''_{kj}$ are fixed numbers) such that $\hat{G}$

is a connected graph. In other words, the condition $G(t) \in \mathbb{G}$ comes to a geometric constraints on the gains ($g_{jk}(t) \in [g'_{jk}; g''_{jk}]$ whenever $g_{jk}(t) \neq 0$) and constant connectivity of the network topology.

Maintaining connectivity (in some sense) is clearly necessary to prevent the network dissemination into separate clusters that do not interact and thus cannot be synchronized. For some classes of agents the constant connectivity assumption may be relaxed, e.g. to the "uniform connectivity" [33], however, extension of our results in this direction in general situation seems to be a non-trivial problem. As discussed in [24], for general unstable agents the uniform connectivity is insufficient for consensus.

In the case of fixed topology we have $\mathbb{G} = \{G_0\}$ so that $\lambda_{2*} = \lambda_2(G_0)$ and $\Delta_* = \max \Delta_j[G_0]$.

We proceed with assumptions concerning the couplings.

Assumption 2: For any $j, k \in \underline{N}$ and $y \in \mathbb{R}^l$ the following anti-symmetry condition holds: $\varphi_{jk}(y) = -\varphi_{kj}(-y)$.

Treating a coupling as an "attracting force" between the agents, Assumption 2 looks like Newton's Third Law. In some applications, e.g. in oscillator networks [12], Assumption 2 holds due to exactly this law.

Although in general the consensus condition (5) says nothing about the behavior of individual state vector y_j, under Assumption 2 the condition (5) implies that $x_j(t) - A^t\tilde{x}_0 \to 0$ as $t \to \infty$ for all j, where $\tilde{x}_0 = \frac{1}{N}\sum_{j=1}^N x_j(0)$. For $A = I_d$ this asymptotical behavior is referred to as the *average consensus* [22]. To establish this, notice that $\sum_{j=1}^N u_j = 0$ due to Assumption 2. By summing up the equations from (3), we see that $\sum_{j=1}^N x_j(t) = A\sum_{j=1}^N x_j(t-1) = NA^t\tilde{x}_0$, thus our claim is evident from (5).

In the present paper we focus on the case when the couplings may be unknown but satisfy a *quadratic constraint* [8]. Given a Hermitian form $F : \mathbb{C}^m \times \mathbb{C}^k \to \mathbb{R}$ let $\mathfrak{S}(F)$ stand for the set of all continuous functions $\varphi : \mathbb{R}^k \to \mathbb{R}^m$ such that $\varphi(0) = 0$ and $F(\varphi(y), y) > 0$ for any $y \neq 0$. In other words, the graph of the function $\varphi = \varphi(y)$ lies strictly inside the cone $F(\varphi, y) > 0$ everywhere except the origin.

Throughout the paper we assume that $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})$ where $\mathcal{F}$ is some known Hermitian form. The consensus criterion should be given in terms of $\mathcal{F}$, the set of uncertain gain matrices $\mathbb{G}$ (in fact, values λ_{2*} and Δ_*) and the coefficients A, B, C, but not the couplings and gains themselves. Such a criterion automatically ensures consensus for all couplings and gains that satisfy aforementioned assumptions, in this sense it may be considered as a criterion of *robust consensus*. Our criterion has especially simple form in the case of SISO agents where quadratic constraints typically shape into *sector* (or *slope*) restrictions (see subsection III-B).

In practice, the class $\mathfrak{S}(\mathcal{F})$ usually contains at least one linear stationary map $y \mapsto Ky$, where K is a constant matrix. If consensus is established *for any* family of couplings $\{\varphi_{jk}(\cdot)\}$ from the set $\mathfrak{S}(\mathcal{F})$, satisfying Assumption 2, and any gain matrix $G(t) \in \mathbb{G}$, then it is established in the special case where $\varphi_{jk}(y) = Ky$ and $G(t) \equiv G_* \in \mathbb{G}$. The consensus condition for such a network is proved analogously to its continuous-time counterpart [14], [22].

Lemma 2: Let $G(t) \equiv G_*$, where the graph $\hat{G}_*$ is undirected and $\varphi_{jk}(y) = Ky$, where $K \in \mathbb{R}^{l \times m}$ is constant. Suppose that the matrix A is not stable. The protocol (4) establishes consensus if and only if the graph $\hat{G}_*$ is connected and the matrix $A - \mu BKC$ is stable for any $\mu \in \sigma(L[G]) \setminus \{0\}$.

In fact, Lemma 2 remains valid for digraph as well, if the connectivity is replaced by the existence of an oriented spanning tree [31]. In the most interesting situation where A has exponentially unstable eigenvalues the matrix $A - \lambda BKC$ becomes unstable both for small $\lambda > 0$ and for $\lambda \to +\infty$. This means that consensus is possible only when $\lambda_2[G_*] \geq \lambda' > 0$ and $\lambda_N[G_*] \leq \lambda''$ for any $G_* \in \mathbb{G}$, where thresholds λ', λ'' depend on A, B, C, K. In other words, robust consensus in the general case implies the existence of positive lower bound for the algebraic connectivity ($\lambda_2(G) \geq \lambda_{2*}$), as well as some upper bound for the maximal Laplacian's eigenvalue λ_N (i.e. $\lambda_N(G) \leq \lambda_{N*}$). Since $\lambda_N(G) \leq 2\max_j \Delta_j(G)$ due to the Gershgorin disk theorem [19], [20], the latter condition on λ_N may be replaced with a bit more conservative yet easily verifiable condition on maximal degree $\Delta(G) \leq \Delta_*$. Assumption 1 says the tightest λ_{2*} and Δ_* are known. However, our criteria in fact retain sufficiency for consensus replacing λ_{2*} and Δ_* with their lower and upper bound, respectively, so there is no need to calculate these values.

III. MAIN RESULTS.

In this section the main result of the paper is presented which establishes sufficient conditions for consensus in general MIMO case. We start with general criterion for MIMO case, given in the subsection III-A. This criterion becomes especially simple in the scalar case where the quadratic constraint typically has the form of the sector inequality (), which case is discussed in the subsection III-B.

A. Consensus criterion for MIMO case.

Throughout this section we assume that the protocol (4) satisfies Assumptions 1,2.

To start with, we introduce some auxiliary notations. Let $W_x(\lambda) = (\lambda I - A)^{-1}B$ and $W_y(\lambda) = CW_x(\lambda)$ stand for the transfer matrices of the plant (3) from u to y, x respectively.

Given a Hermitian form $\mathcal{F}(\varphi, y)$ as follows

$$\mathcal{F}(\varphi, y) = Re(\varphi^* q y) - y^* Q y - \varphi^* R \varphi,\ y \in \mathbb{C}^l, \varphi \in \mathbb{C}^m \quad (7)$$

(where $q \in \mathbb{R}^{m \times l}$, $Q = Q^* \in \mathbb{R}^{l \times l}$, $R = R^* \in \mathbb{R}^{m \times m}$), we introduce a function $\Pi_{\mathcal{F}} : \mathbb{C}^{l \times m} \to \mathbb{C}^{m \times m}$ as follows

$$\Pi_{\mathcal{F}}(\Lambda) := q\Lambda + \Lambda^* q^* + \lambda_{2*}\Lambda^* Q \Lambda + \frac{1}{2\Delta_*}R, \qquad (8)$$

where λ_{2*}, Δ_* are defined in (6). Now we are in position to formulate our main result.

Theorem 1: Suppose that Assumptions 1 and 2 hold and $\varphi_{jk} \in \mathfrak{S}(\mathcal{F})\ \forall j, k$ where $\mathcal{F}$ is a quadratic form (7) with $Q \geq 0, \Gamma \geq 0$. Assume also that

(a) There exist matrix $K \in \mathbb{R}^{m \times l}$ and matrix of coupling gains $G_* \in \mathbb{G}$ such that $\mathcal{F}(Ky, y) > 0\ \forall y \neq 0$ and $A - \mu BKC$ is a stable matrix for any $\mu \in \sigma(L(G_*)) \setminus \{0\}$;

(b) for any $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\det(\lambda I - A) \neq 0$ one has $\Pi_{\mathcal{F}}(W_y(\lambda)) \geq 0$.

Then the protocol (4) establishes the consensus. Moreover, if $\varepsilon > 0$ exists such that $\Pi_{\mathcal{F}}(W_y(\lambda)) \geq \varepsilon W_x(\lambda)^* W_x(\lambda)$ whenever $|\lambda| = 1$, it establishes the exponential consensus.

The proof of Theorem 1 retraces arguments from [25], the details may be found in a companion paper [28].

By Theorem 1, conditions (a) and (b) in fact imply the robust consensus in the sense that (5) holds for any uncertain couplings and gains satisfying the aforementioned assumptions. In practice (a) is almost unavoidable for such a robust consensus. The existence of a matrix K such that the map $y \mapsto Ky$ belongs to $\mathfrak{S}(\mathcal{F})$ is typically a non-restrictive assumption which is fulfilled, for instance, for scalar case and sector inequalities (see subsection III-B). By Lemma 2, robust consensus implies that condition (a) eventually holds for any such matrix K.

Remark 1: As discussed in the foregoing, for any matrix $G \in \mathbb{G}$ one has $\lambda_2(G) \geq \lambda_{2*}$ and $\lambda_N(G) \leq 2\Delta_*$. Therefore, condition (a) may be replaced with a stronger condition as follows, retaining the sufficiency of consensus: there exists matrix $K \in \mathbb{R}^{m\times l}$ such that $\mathcal{F}(Ky, y) > 0\ \forall y \neq 0$ and $A - \mu BKC$ is a stable matrix for any $\mu \in [\lambda_{2*}; 2\Delta_*]$.

The condition (b) in Theorem 1 explicitly involves values λ_{2*} and Δ_* which depend on the class $\mathbb{G}$. However, since by assumption $Q \geq 0$, formal replacement of λ_{2*} by its lower estimate retains sufficiency for consensus, and the same is true for replacement of Δ_* with its upper estimate. This observation is useful whenever the exact computation of those values is troublesome, in particular, many available constructive estimates of the algebraic connectivity, may be put in use [19]. Furthermore, the value of λ_{2*} is unimportant if $Q = 0$ and if $\Gamma = 0$ then Δ_* is not involved in (b).

As will be demonstrated below for the scalar case, the result of Theorem 1 may be considered as an extension of the multi-variable circle criterion for discrete-time systems, known also as the Tsypkin stability criterion [9], [38]. Under additional assumption of constant interaction topology (the cardinality of $\mathbb{G}$ equals to 1) the result of Theorem 1 may be extended to some wider class of constraints, involving delays [28]. Moreover, applying techniques from the recent paper [29], a counterpart of the discrete-time Popov criterion (usually referred to as the Jury-Lee stability criterion [9], [10]) may be derived. In this paper we confine ourselves to the case of switching topology.

B. Consensus for scalar sectorial couplings

In this paragraph we apply the result of Theorem 1 to SISO agents coupled by scalar maps φ_{jk}, that are restricted by the sector with known slopes [8], [11]. Throughout this subsection $k = l = 1$ and we assume that $\varphi_{jk} \in S[\alpha; \beta]$ where $S[\alpha; \beta]$ is a set of all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) \equiv 0$ and

$$\alpha < \frac{\varphi(\sigma)}{\sigma} < \beta \quad \sigma \neq 0 \tag{9}$$

fIt follows from (9) that the graph of the function $\varphi = \varphi(\sigma)$ lies strictly between the lines $\varphi = \alpha\sigma$ and $\varphi = \beta\sigma$ everywhere except for the origin.

One can easily transform the sector restrictions (9) into quadratic constraint, introducing the constants

$$\gamma = \frac{1}{\beta + \alpha} \geq 0, \quad \delta = \frac{\alpha}{1 + \alpha\beta^{-1}} \geq 0, \tag{10}$$

and the Hermitian forms as follows:

$$\mathcal{F}_{\alpha;\beta}(\varphi, y) := \varphi y - \delta y^2 - \gamma\varphi^2,\ \varphi, y \in \mathbb{R} \tag{11}$$

$$\Pi_{\alpha;\beta}(\lambda) = Re\,\lambda + \lambda_{2*}\delta|\lambda|^2 + \frac{\gamma}{2\Delta_*},\ \lambda \in \mathbb{C}. \tag{12}$$

Lemma 3: $S[\alpha; \beta] = \mathfrak{S}(\mathcal{F}_{\alpha;\beta})$ and $\Pi_{\mathcal{F}_{\alpha;\beta}} \equiv \Pi_{\alpha;\beta}$.

Lemma 3 is proved by a straightforward computation. As a consequence, we obtain the following consensus criterion for the scalar case.

Theorem 2: Suppose that Assumptions 1 and 2 are valid, $\varphi_{jk} \in S[\alpha; \beta]\ \forall j, k$ where $0 \leq \alpha < \beta \leq \infty$, and the following two claims hold:

(a) There exists $\varkappa \in (\alpha; \beta)$ and $G_* \in \mathbb{G}$ such that the matrix $A - \mu\varkappa BC$ is stable for any $\mu \in \sigma(L(G_*)) \setminus \{0\}$;
(b) for any $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$ and $\lambda \notin \sigma(A)$, one has $\Pi_{\alpha;\beta}(W_y(\lambda)) \geq 0$.

Then the protocol (4) provides consensus. If additionally $\varepsilon > 0$ exists such that $\Pi_{\alpha;\beta}(W_y(\lambda)) \geq \varepsilon|W_x(\lambda)|^2$, it establishes the exponential consensus.

Proof: This result follows from Theorem 1 and Lemma 3. Indeed, (a) and (b) coincide with the corresponding conditions from Theorem 1, applied for $\mathcal{F} := F_{\alpha,\beta}$. ■

Like in the continuous time case [24], [27], Theorem 2 may be considered as a networked version of the well-known circle criterion for stability of discrete-time systems, known also the Tsypkin criterion [38]. The frequency-domain condition (b) in Theorem 2 may be rewritten in the form of "circle criterion": the Nyquist curve $\{W_y(\lambda) : |\lambda| = 1\}$ lies outside the set $\mathcal{D}$ which is the open half-plane $\{z \in \mathbb{C} : Re\,z < -\frac{\gamma}{2\Delta_*}\}$ for $\alpha = 0$ and the disk $\mathcal{D} = \{z \in \mathbb{C} : |z - z_0| < \rho_0\}$ for $\alpha > 0$, whose center and radius are

$$z_0 = -\frac{1}{2\delta\lambda_{2*}}, \quad \rho_0 = \frac{1}{2\delta\lambda_{2*}}\sqrt{1 - \frac{2\lambda_{2*}\delta\gamma}{\Delta_*}}.$$

Retracing arguments from [24, Appendix A], it may be shown that the classical Tsypkin crterion for stability of discrete time Lurie systems [9], [13], [38] is equivalent to Theorem 2 applied to a network with $N = 2$ agents.

IV. Examples.

Now we illustrate the potential of Theorem 1 by showing that its specifications for agents with special dynamics provide improvements of recent results in the area.

A. First-order agents with input delay

In this subsection, we consider the team of agents modeled by the following equations:

$$y_j(t + 1) = y_j(t) + u_j(t - T) \in \mathbb{R},\ t = 0, 1, 2, \ldots \tag{13}$$

Here $y_j(t)$ and $u_j(t)$ stand, respectively, for the output and control of the j-th agent and known integer number $T \geq 0$ stands for the input *delay*. In the undelayed case ($T = 0$) the equation (13) describes a conventional first-order integrator. We are interested in the conditions which guarantee the protocol (4) to establish consensus between the agents (13).

Most existing results on consensus among delayed discrete-time integrator agents deal with the case of so called *communication* delay which affects only the neighbor's data, whereas the agent has direct access to its own state [1], [3], [4], [18]. In this case, even time-varying delays do not violate the consensus provided that they remain bounded. This is not the case if own state of the agent is delayed, as implied, for example, by the dynamics (13),(4). The problem of consensus with self-delays was mainly addressed in the continuous-time case [16], [36] with the help of the Lyapunov-Krasovskii method. Discrete-time agents with self-delays were considered only in the case of time-invariant topology and delays [37] and special structure of delay [42], and both of these results addressed linearly coupled networks. The following theorem gives sufficient conditions for consensus under nonlinear couplings in (4).

Theorem 3: Suppose that Assumptions 1 and 2 hold and $\varphi_{jk} \in S[0; \beta]$ for some $\beta < \infty$. If the inequality

$$(2T + 1)\beta\Delta_* < 1 \tag{14}$$

is valid, the protocol (4) establishes consensus. If $\varphi_{jk} \in S[\alpha; \beta]$ for some $\alpha > 0$, it establishes exponential consensus.

Proof: The system (13) may be rewritten in the state-space form by introducing the state vector $x(t) = (y(t), y(t-1), \dots, y(t-T+1), u(t-1), \dots, u(t-T+1))^T$. As follows from [37, Theorem 2], the protocol (4) with linear couplings $\varphi_{jk}(y) = \mu y$ ($\mu \in (0; \beta]$) and constant topology $G(t) \equiv G_* \in \mathbb{G}$ (in particular, $\max_j \Delta_j(G) \leq \Delta_*$) establishes consensus if (14) holds or, equivalently, condition (a) in Theorem 2 holds for any $\kappa \in (0; \beta)$. A straightforward computation shows that for $\lambda = e^{i\omega}$, $\omega \in [-\pi; \pi]$, one has

$$Re\, W_y(\lambda) = Re\, \frac{\lambda^{-T}}{\lambda - 1} = \frac{Re[\lambda^{-T-1} - \lambda^{-T}]}{|\lambda - 1|^2} = \\ = -\frac{\sin\frac{(2T+1)\omega}{2}}{2\sin\frac{\omega}{2}} \geq -\frac{2T+1}{2}$$

(the latter inequality was proved e.g. in [37]). In particular, $\Pi_{0;\beta}(W_y(\lambda)) \geq \frac{1}{2\beta\Delta_*} - \frac{2T+1}{2}$ and hence (14) implies the condition (b) in Theorem 2, moreover, $\Pi_{0;\beta}(W_y(\lambda)) \geq \varepsilon_0$ for ε_0 sufficiently small. This proves consensus. For any $\alpha \in (0; \beta)$ one has $\Pi_{\alpha;\beta}(W_y(\lambda)) = \Pi_{0;\beta}(W_y(\lambda)) + \delta|W_y(\lambda)|^2 \geq \varepsilon_0 + \delta/|\lambda - 1|^2$, where δ is defined in (10). Hence $\Pi_{\alpha;\beta}(W_y(\lambda)) \geq \varepsilon|W_x(\lambda)|^2$ for $\varepsilon > 0$ being small, which proves exponential consensus if $\varphi_{jk} \in S[\alpha; \beta]$. ■

In the special case where $G(t) = const$ and $\varphi_{jk}(y) = y$ (thus $\beta = 1$) Theorem 3 was proved in [37]. Moreover, this result holds for directed graph as well [37, Theorem 2]. For undirected networks, the bound on delay (14) under which the consensus is established may be tightened [37, Theorem 1] and extended to the case of heterogeneous delays. The approach from [37] is based on frequency-domains techniques and not applicable to nonlinear couplings and time-varying graphs, which are the main concern of the present paper. Theorem 3 improves the recent result of [26, Theorem 2], which is based on ideas from [23] and implies consensus under more conservative condition $2(T+1)\beta\Delta_* < 1$ (in the notations of [26], γ corresponds to our β) in the special case of homogeneous constant delays.

B. Second-order consensus with velocity measurements

In the present subsection we consider a team of second-order agents obtained via the forward difference approximation of continuous double integrator agents [30].

$$y_j(t+1) = y_j(t) + v_j(t),\ v_j(t+1) = v_j(t) + \xi_j(t). \tag{15}$$

Assuming the agents have access to their absolute velocities, the following consensus protocol was proposed in [18], [30]:

$$\xi_j(t) = -2\eta v_j(t) + \sum_{k=1}^{N} g_{jk}(t)(y_k(t) - y_j(t)),$$

and its convergence was proven by using results from theory of stochastic matrices. Below we examine convergence of more general nonlinear protocols

$$\xi_j(t) = -2\eta v_j(t) + \sum_{k=1}^{N} g_{jk}(t)\varphi_{jk}(y_k(t) - y_j(t)). \tag{16}$$

It is easily seen that the closed-loop networked system (15),(16) coincides with a network of agents

$$y_j(t+1) = y_j(t) + v_j(t),\ v_j(t+1) = (1 - 2\eta)v_j(t) + u_j(t), \tag{17}$$

which apply the protocol (4). This allows to apply Theorem 2, which results in the following consensus criterion.

Theorem 4: Suppose that Assumptions 1 and 2 hold and $\varphi_{jk} \in S[0; \beta]$ for some $\beta < \infty$. If $\eta \in (0; 1)$ and

$$\beta\Delta_* \leq \frac{2\eta^2}{1 + \eta}, \tag{18}$$

then consensus among the agents (15) is established.

Remark 2: The inequality (18) holds if $\sqrt{\beta\Delta_*} < \eta < 1$, which condition is sufficient for consensus with linear couplings $\varphi_{jk}(y) = \theta y$, $\theta \in (0; \beta)$ [30]. The result in [30] works for directed topology which is assumed to be only jointly connected. Although our result is valid only for undirected and connected graphs, it offers, on the positive side, less restrictive consensus condition (18) instead of the inequality $\sqrt{\beta\Delta_*} < \eta$ and deals with *nonlinear* protocols.

Proof: As follows from the result of [30] just mentioned, for sufficiently small $\varepsilon > 0$ and any $G \in \mathbb{G}$ the protocol (4) with $\varphi_{jk}(y) = \varepsilon y$ and $G(t) \equiv G$ establishes consensus, which proves condition (a) in Theorem 2. We are going to verify (b) for the agent (17). Denoting $\theta := 1 - 2\eta$

and taking $\lambda = e^{i\omega}$, $\omega \in [-\pi;\pi]$, it is easily shown that

$$Re\,W_y(\lambda) = \frac{Re[(\bar\lambda - 1)(\bar\lambda - \theta)]}{|\lambda - 1|^2|\lambda - \theta|^2} =$$

$$= \frac{\cos 2\omega - \cos\omega - \theta(\cos\omega - 1)}{2(1-\cos\omega)(1+\theta^2 - 2\theta\cos\omega)} =$$

$$= -\frac{\theta - \frac{\sin\frac{3\omega}{2}}{\sin\frac{\omega}{2}}}{2(1+\theta^2-2\theta\cos\omega)} \geq$$

$$\geq \frac{\theta - 3}{2(1+\theta^2-2\theta\cos\omega)} \geq \frac{-1-\eta}{(1-\theta)^2} = -\frac{1+\eta}{4\eta^2}.$$

Therefore, $\Pi_{0;\beta}(W_y(\lambda)) = 1/(2\beta\Delta_*) - (1+\eta)/(4\eta^2) \geq 0$, that is, condition (b) is valid, if (18) holds. ■

V. Conclusion

State consensus among identical linear MIMO discrete-time agents was examined in the case where the interaction topology may be uncertain and switching but assumed to preserve its connectivity. The couplings among the agents are uncertain as well and satisfy the conventional quadratic constraints. A new criterion for robust output consensus is established which may be considered as the networked analogue of the circle criterion for discrete-time systems (known also as the Tsypkin criterion) in absolute stability theory. Its extension on the leader-following formation control, reference tracking consensus, containment control and nonlinear agents of Lurie type are topics of ongoing research. In the case of time-invariant topology, the conservatism of the proposed criterion may be reduced by using more intricate quadratic constraints, involving delays. This approach leads to an analogue of the Jury-Lee criterion (Popov-type criterion for stability of discrete-time systems). Some initial steps in this direction were done in a companion paper [28].

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