



Figure 1: An illustrative graph in $\mathcal{N}(2, 2)$ with 8 hubs.

$G \in \mathcal{N}(C_1, C_2, \dots, C_n)$, but for any $e \in E$, $G \setminus \{e\} \notin \mathcal{N}(C_1, C_2, \dots, C_n)$. It then follows from the fact that every graph in $\mathcal{N}(C_1, C_2, \dots, C_n)$ has at least one minimal subgraph that

$$\mathcal{H}(C_1, C_2, \dots, C_n) = \sup_{\substack{G \in \mathcal{N}(C_1, C_2, \dots, C_n) \\ G \text{ is minimal}}} \mathcal{H}(G).$$

The vertex-connectivity version of the classical Menger’s theorem [4] states that for a network with one pair of source S and sink R with the minimum vertex-cut between them being C , there exist C vertex-disjoint paths connecting S and R , which immediately implies that $\mathcal{H}(C) = 0$ for any given C . The above-mentioned finiteness theorem states that for any given $G \in \mathcal{N}(C_1, C_2, \dots, C_n)$, one can always find a subgraph \widehat{G} of G such that $\mathcal{H}(\widehat{G})$ is upper bounded by a constant, which is independent of the choice of G . In some sense, this result can be viewed as a generalization of the vertex-connectivity version of Menger’s theorem.

Mathematically, the proposed problem of computing $\mathcal{H}(C_1, C_2, \dots, C_n)$ is a natural combinatorial optimization problem. On a more practical side, hubs in networks naturally correspond to more costly vertices. For instance, in a transportation network, as opposed to “relaying” vertices with degree 2, hubs may have to be better equipped for traffic scheduling; for this reason, when designing the route map, an airline may need to avoid running too many airline hubs to reduce the cost. So, as might be expected, $\mathcal{H}(C_1, C_2, \dots, C_n)$ is of significance to cost-minimizing resource allocation in transportation networks. Roughly speaking, our results suggest that for some given flow demand constraints, not too many hubs are needed in a network.

To the best of our knowledge, the proposed problem of computing or estimating \mathcal{H} has not yet been examined previously and there is little related work in the vast literature of graph theory. On the other hand, to a great extent, this work is motivated by the study of network encoding complexity (see [3] and references therein), where the number of encoding vertices in directed networks is of primary concern. Moreover, our approaches to tackle the problem are influenced by those in network encoding complexity theory, particularly, to a greater extent, those in [2, 5].

The contribution of this paper can be summarized as follows:

- We give necessary and sufficient conditions for a graph being minimal in $\mathcal{N}(C_1, C_2)$.

- We introduce the notion of a representation of a graph in $\mathcal{N}(C_1, C_2)$ and we present the structural decomposition theorem for representations of minimal graphs in $\mathcal{N}(C_1, C_2)$.
- We introduce a novel path-searching algorithm, the analysis of which will aptly produce an upper bound on $\mathcal{H}(C_1, C_2)$ for any given C_1, C_2 .
- We derive the value of $\mathcal{H}(C_1, C_2)$, which is a primary result in this paper.
- We establish the finiteness (another key result in this paper) of $\mathcal{H}(C_1, C_2, \dots, C_n)$, $n \geq 3$, through a recursive bounding argument.
- We also derive the values of \mathcal{H} for some special parameters

References

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