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Impulse Control of the Beneš Process*

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***Abstract*— This paper combines linear programming and nonlinear optimization to demonstrate a new approach to the solution of impulse control problems. It examines the optimal process arising from the Bounded Follower Problem of Beneš et al. [1] in which one seeks to minimize the discounted second moment of the difference between a Brownian motion process W and its follower, which is limited to pursuing W using bounded velocity. Additional control of the optimal process is included in the model through the imposition of costly instantaneous shifts of the process. By first restricting the set of admissible impulse policies, this paper identifies two linear programs over spaces of measures into which the stochastic problem is imbedded, thus providing lower bounds on the optimal value. The first problem naturally leads to a nonlinear optimization problem which one is able to solve. The optimizers of this nonlinear function are then used to specify an impulse policy whose expected cost, for some initial positions, equals the lower bound and hence this policy is optimal. Solving the dual of the second linear program determines the optimal value for every initial position, again over the restricted set of policies. A stochastic argument then demonstrates that the expected cost of each admissible impulse control policy is bounded below by the optimal expected cost over the restricted class of policies.**

I. INTRODUCTION

The classic paper of Beneš et al. [1] introduced several stochastic control problems in which the optimal control is of bang-bang type. This paper is concerned, in particular, with the Bounded Velocity Follower problem in which one seeks to select a process of bounded variation having bounded velocity so as to minimize the discounted second moment of the difference between a Brownian motion process and the bounded variation process. The resulting difference is a drifted Brownian motion process. The optimal control policy is proven to switch instantaneously between the minimum drift rate and the maximum drift rate as the difference repeatedly crosses a threshold. For the purposes of this paper, the drift rate is restricted to the interval $[-\mu, \mu]$ for some positive constant μ and the optimally controlled process X, to which we refer as the Beneš process, is a solution of the stochastic differential equation

$$dX(t) = -\mu\,\mathrm{sgn}(X(t))\,dt + \sigma\,dW(t), \quad X(0) = x_0, \quad (1)$$

in which W is a standard Brownian motion process and $\mu, \sigma > 0$. Let $\mathcal{F}$ denote the natural filtration of W.

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Our variation on this problem is to add additional control options on the process X through the inclusion of impulses. An impulse control policy consists of a pair of sequences $(\tau, Y) := \{(\tau_k, Y_k) : k \in \mathbb{N}\}$ in which τ_k is the $\{\mathcal{F}_t\}$-stopping time of the kth impulse and the $\mathcal{F}_{\tau_k}$-measurable variable Y_k gives the kth impulse size. The sequence $\{\tau_k : k \in \mathbb{N}\}$ is required to be non-decreasing, a natural assumption in that intervention $k+1$ must occur no earlier than intervention k. For a policy (τ, Y), the resulting impulse-controlled process (also denoted by X) then satisfies

$$\begin{aligned} X(t) = x_0 - \mu \int_0^t \mathrm{sgn}(X(s))\,ds + \sigma\,W(t) \\ + \sum_{k=1}^{\infty} I_{\{\tau_k \le t\}} Y_k. \end{aligned} \quad (2)$$

In contrast with the Beneš problem for which no cost is accrued by either the choice of drift rate or the action of switching between rates, each impulse is costly. Let (τ, Y) be an impulse control policy. Define $c_0(x) = x^2$. Let $k_1 > 0$ denote the fixed costs incurred for each intervention and let $k_2 \ge 0$ be a cost proportional to the size of the intervention. Define the impulse cost function $c_1(y, z) = k_1 + k_2|z - y|$, in which y denotes the pre-jump location of X (typically far from 0) and z denotes the post-jump location of X which is thought to be close to 0. Let $\alpha > 0$ denote the discount rate. The goal of this paper is to find an optimal impulse policy $(\tau^*, Y^*) = \{(\tau_k^*, Y_k^*) : k \in \mathbb{N}\}$ so as to minimize the expected discounted cost

$$\begin{aligned} J(\tau, Y; x_0) := \mathbb{E}_{x_0}\Bigg[\int_0^{\infty} e^{-\alpha s} c_0(X(s))\,ds \\ + \sum_{k=1}^{\infty} e^{-\alpha \tau_k} I_{\{\tau_k < \infty\}} c_1(X(\tau_k-), X(\tau_k)\Bigg]. \end{aligned} \quad (3)$$

The controller must balance the desire to keep the process X near 0 so as to have a small second moment against the desire to limit the number and/or sizes of interventions so as to have a small impulse cost. Since the goal is to minimize the objective function, impulse control policies having $J(\tau, Y; x_0) = \infty$ are undesirable. We therefore restrict attention to the impulse policies for which $J(\tau, Y; x_0)$ is finite. Denote this class of *admissible* controls by $\mathcal{A}$.

We make five important observations about impulse policies. Firstly, “0-size impulses” which do not change the state only increase the cost so can be excluded from consideration. Secondly, the symmetry of the dynamics and costs means that any impulse (τ_k, Y_k) which would cause $\mathrm{sgn}(X(\tau_k)) = -\mathrm{sgn}(X(\tau_k-))$ on a set of positive probability will have no greater cost (smaller cost when $k_2 > 0$) by replacing the impulse with one for which $\tilde{X}(\tau_k) = \text{sgn}(X(\tau_k-))|X(\tau_k)|$. Thus we can also restrict analysis to those policies for which all impulses keep the process on the same side of 0. Next, any policy (τ, Y) with $\lim_{k\to\infty} \tau_k =: \tau_\infty < \infty$ on a set of positive probability will have infinite cost so for every admissible policy $\tau_k \to \infty$ $a.s.$ as $k \to \infty$. Fourthly, let (τ, Y) be a policy for which there is some k such that $\tau_k = \tau_{k+1}$ on a set of positive probability. Again due to the presence of the fixed intervention cost k_1, the total cost up to time τ_{k+1} will be at least $k_1\mathbb{E}[e^{-\alpha\tau_k} I_{\{\tau_k=\tau_{k+1}\}}]$ smaller by combining these interventions into a single intervention on this set. Hence we may restrict policies to those for which $\tau_k < \tau_{k+1}$ $a.s.$ for each k.

The final observation is similar. Suppose (τ, Y) is a policy such that on a set G of positive probability $\tau_k < \infty$ and $|X(\tau_k)| > |X(\tau_k-)|$ for some k. Consider a modification of this impulse policy and resulting process $\tilde{X}$ which simply fails to implement this impulse on G. Define the stopping time $\sigma = \inf\{t > \tau_k : |X(t)| \leq |\tilde{X}(t)|\}$. Notice that the running costs accrued by $\tilde{X}$ over $[\tau_k, \sigma)$ are smaller than those accrued by X. Finally, at time σ, introduce an intervention on the set G which moves the $\tilde{X}$ process so that $\tilde{X}(\sigma) = X(\sigma)$. This intervention will incur a cost which is smaller than the cost for the process X at time τ_k. As a result, we may restrict the impulse control policies to those for which every impulse decreases the distance of the process from the origin.

Similar type of problems have been extensively investigated in the literature. An incomplete list includes the now classical works on general stochastic impulse problems [2], [5], [8], [12] as well as their applications in various areas such as portfolio optimization, inventory control, risk management, control of a dam and exchange rate intervention [3], [4], [9], [10], [11]. The standard approach toward solving impulse control problems is to solve associated quasi-variational inequalities for the value function.

This paper extends a linear programming approach used on optimal stopping problems [6]. The stochastic problem is imbedded in several infinite-dimensional linear programs over a space of expected discounted measures. A simple observation about the first linear program requires the optimization of a nonlinear function. The optimizers of this function are used to determine an optimal impulse control policy for the original stochastic problem *for some initial positions* of the process. Another linear program and its dual are used to obtain an optimal policy for all initial positions.

II. RESTRICTED PROBLEM

The solution of the impulse control problem is obtained by first considering a subclass of the admissible impulse control pairs.

Condition II.1. *Let $\mathcal{A}_1 \subset \mathcal{A}$ be those policies (τ, Y) such that the resulting process X is bounded; that is, for $(\tau, Y) \in \mathcal{A}_1$, there exists some $M < \infty$ such that $|X(t)| \leq M$ for all $t \geq 0$.*

Note that for each $M > 0$, any impulse control which has the process jump closer to 0 whenever $|X(t-)| = M$ is in the class $\mathcal{A}_1$ so this collection is non-empty. The bound is not required to be uniform for all $(\tau, Y) \in \mathcal{A}_1$. The restricted impulse control problem is one of minimizing $J(\tau, Y; x_0)$ over all policies $(\tau, Y) \in \mathcal{A}_1$.

A. First Linear Program

To determine the first linear program, we identify two particular functions p_0 and g_0. Recall the generator A of the Beneš process is

$$Af(x) = \tfrac{\sigma^2}{2} f''(x) - \mu \text{sgn}(x) f'(x).$$

The function p_0 is a solution of the eigenvalue equation $\alpha f - Af = 0$ with the properties that p_0 is symmetric, strictly decreasing for $x < 0$ and strictly increasing for $x > 0$. Let $\gamma_1 := \frac{\mu}{\sigma^2} - \sqrt{(\frac{\mu}{\sigma^2})^2 + \frac{2\alpha}{\sigma^2}} < 0$ and $\gamma_2 := \frac{\mu}{\sigma^2} + \sqrt{(\frac{\mu}{\sigma^2})^2 + \frac{2\alpha}{\sigma^2}} > 0$ be the solutions of the characteristic equation $\frac{\sigma^2}{2}\gamma^2 - \mu\gamma - \alpha = 0$ which arises when $x > 0$. The solutions to the characteristic equation when $x < 0$ are $-\gamma_2 < 0 < -\gamma_1$. A C^2-function p_0 with the desired properties is

$$p_0(x) = \begin{cases} -\gamma_1 e^{-\gamma_2 x} + \gamma_2 e^{-\gamma_1 x}, & \text{for } x \leq 0, \\ \gamma_2 e^{\gamma_1 x} - \gamma_1 e^{\gamma_2 x}, & \text{for } x \geq 0. \end{cases}$$

For each $(\tau, Y) \in \mathcal{A}_1$, the process X is bounded so applying the general Dynkin's formula and letting $t \to \infty$, we obtain the identity

$$\begin{aligned} p_0(x_0) &= \mathbb{E}_{x_0}\left[\int_0^\infty e^{-\alpha s}[\alpha p_0 - Ap_0](X(s))\,ds\right] \\ &\quad - \mathbb{E}_{x_0}\left[\sum_{k=1}^\infty I_{\{\tau_k<\infty\}} e^{-\alpha\tau_k} Bp_0(X(\tau_k-), X(\tau_k))\right], \end{aligned} \tag{4}$$

in which $Bp_0(y, z) = p_0(z) - p_0(y)$ denotes the change in the function p_0 due to a jump from y to z. Observe that the choice of p_0 results in the integrand of the first expectation being 0 so (4) simplifies.

Turning to the second function g_0. Let X be the Beneš process having no impulses and define g_0 by

$$g_0(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\alpha s} c_0(X(s))\,ds\right]. \tag{5}$$

It is not difficult to verify (see e.g., [7, Proposition 3.1], though there is a typographical error in that formula which is corrected in the formula below) that

$$g_0(x) = \tfrac{1}{\alpha}x^2 - \tfrac{2\mu}{\alpha^2}|x| + \tfrac{2\mu^2}{\alpha^3} + \tfrac{\sigma^2}{\alpha^2} + \tfrac{2\mu}{\alpha^2\gamma_1} e^{\gamma_1|x|} \tag{6}$$

and that $g_0 \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} - \{0\})$ as well as g_0 being a solution of $\alpha f - Af = c_0$ on $\mathbb{R} - \{0\}$.

A reformulation of the objective function is also needed. Let $(\tau, Y) \in \mathcal{A}_1$ be given. Then using g_0 and letting $t \to \infty$, the general Dynkin's formula results in

$$\begin{aligned} g_0(x_0) + \mathbb{E}_{x_0}\left[\sum_{k=0}^{\infty} I_{\{\tau_k<\infty\}} e^{-\alpha\tau_k} Bg_0(X(\tau_k-), X(\tau_k))\right] \\ = \mathbb{E}_{x_0}\left[\int_0^\infty e^{-\alpha s}[\alpha g_0(X(s)) - Ag_0(X(s))]\, ds\right] \\ = \mathbb{E}_{x_0}\left[\int_0^\infty e^{-\alpha s} c_0(X(s))\, ds\right] \end{aligned} \tag{7}$$

in which $Bg_0(y, z) = g_0(z) - g_0(y)$ and the transversality condition $\lim_{t\to\infty} \mathbb{E}_{x_0}[e^{-\alpha t} g_0(X(t))] = 0$ follows from the boundedness of X. The objective function $J(\tau, Y; x_0)$ of (3) then takes the form

$$\begin{aligned} g_0(x_0) + \mathbb{E}_{x_0}\Bigg[\sum_{k=0}^{\infty} I_{\{\tau_k<\infty\}} e^{-\alpha\tau_k} [c_1(x(\tau_k-), X(\tau_k)) \\ + Bg_0(X(\tau_k-), X(\tau_k))\Bigg]. \end{aligned} \tag{8}$$

Define the discounted expected occupation measure μ_0 and the discounted impulse measure μ_1 such that for each $G \subset \mathbb{R}$ and $G_1 \subset \mathbb{R}^2$,

$$\mu_0(G) = \mathbb{E}_{x_0}\left[\int_0^\infty e^{-\alpha s} I_G(X(s))\, ds\right] \tag{9}$$

$$\mu_1(G_1) = \mathbb{E}_{x_0}\left[\sum_{k=0}^{\infty} I_{\{\tau_k<\infty\}} e^{-\alpha\tau_k} I_{G_1}(X(\tau_k-), X(\tau_k))\right].$$

Notice that the total mass of μ_0 is $1/\alpha$ while μ_1 is a finite measure since $J(\tau, Y; x_0)$ is finite. The measure μ_1 is used immediately whereas μ_0 will only be used in Section II-C.

Rewriting the objective function (8) and Dynkin's formula (4) involving p_0 in terms of these measures imbeds the impulse control problem in the "linear" program

$$\begin{cases} \text{Min.} & g_0(x_0) + \displaystyle\int [c_1 + Bg_0]\, d\mu_1 \\ \text{S.t.} & \displaystyle\int (-Bp_0)\, d\mu_1 = p_0(x_0). \end{cases} \tag{10}$$

In defining the program (10), the affine term $g_0(x_0)$ in the objective function is common for all $(\tau, Y) \in \mathcal{A}_1$ and may therefore be ignored in solving (10). It is important to remember that this term must be included to determine the optimal value of the impulse control problem.

Let $V_1(x_0)$ denote the optimal value of the impulse control problem over the class $\mathcal{A}_1$ of restricted policies and let $V_{lp1}(x_0)$ denote the optimal value of the (10). The argument above showing the imbedding of the values of the objective function (8) corresponding to policies $(\tau, Y) \in \mathcal{A}_1$ in the linear program immediately implies the following relation.

Theorem II.2. $V_{lp1}(x_0) \leq V_1(x_0)$.

B. Nonlinear Optimization

Recall, the admissible impulse policies can be (and are) limited to those for which impulses move X closer to the origin. As a result, the integrand $-Bp_0 > 0$ and the constraint of (10) implies that the feasible measures μ_1 of (10) are those for which $-Bp_0/p_0(x_0)$ is a probability density. For a feasible μ_1, let $\tilde{\mu}_1$ be the corresponding probability measure having density $\frac{-Bp_0}{p_0(x_0)}$. Thus we can write the objective function as

$$\int [c_1 + Bg_0]\, d\mu_1 = \left(\int \frac{c_1 + Bg_0}{-Bp_0}\, d\tilde{\mu}_1\right) p_0(x_0).$$

Since the goal is to minimize the cost, a lower bound is given by the minimal value of F scaled by the constant $p_0(x_0)$, where

$$F(y, z) := \frac{c_1(y, z) + Bg_0(y, z)}{-Bp_0(y, z)}. \tag{11}$$

Moreover, should the infimum be attained at some pair (y_*, z_*), then the probability measure $\tilde{\mu}_1(\cdot)$ putting unit point mass on (y_*, z_*) would achieve the lower bound and identify an optimal μ_1 measure for the linear program (10). To solve the stochastic problem, one would need to connect the measure μ_1 back to an admissible impulse control policy in the class $\mathcal{A}_1$ in such a way that the resulting μ_1 measure would be given by (9).

Proposition II.3. *There exists pairs (y_*, z_*) and $(-y_*, -z_*)$ such that*

$$F(y_*, z_*) = F(-y_*, -z_*) = \inf_{(y.z):|z|\leq|y|} F(y, z). \tag{12}$$

Proof: Recall we only need to examine the cases in which the impulses move the process closer to the origin. Observe F is symmetric through 0 in that $F(-y, -z) = F(y, z)$ so it is sufficient to analyze F on the domain $0 \leq z < y$.

The function F is

$$\frac{k_1 + k_2(y - z) + \frac{z^2 - y^2}{\alpha} + \frac{2\mu}{\alpha^2}(y - z) + \frac{2\mu}{\alpha^2\gamma_1}(e^{\gamma_1 z} - e^{\gamma_1 y})}{\gamma_2(e^{\gamma_1 y} - e^{\gamma_1 z}) - \gamma_1(e^{\gamma_2 y} - e^{\gamma_2 z})}.$$

There exists some pairs (y, z) for which $F(y, z) < 0$ since the difference of the quadratic terms is negative and can dominate the constant and linear terms in the numerator while the exponential terms also provide a negative contribution since $\gamma_1 < 0$.

We need to determine the limiting values as $(y, z) \to \infty$ with $z < y$. When z remains bounded and $y \to \infty$, the term $e^{\gamma_2 y}$ dominates and thus

$$\lim_{(y,z)\to\infty} F(y, z) = 0.$$

Now consider the situation in which $z \to \infty$ and hence $y \to \infty$. Then the difference $e^{\gamma_1 z} - e^{\gamma_1 y} \to 0$ since $\gamma_1 < 0$ so these terms in both the numerator and denominator of F can be ignored. We therefore seek to analyze the asymptotics of

$$G(y, z) = \frac{k_1 + k_2(y - z) + \frac{z^2 - y^2}{\alpha} + \frac{2\mu}{\alpha^2}(y - z)}{(-\gamma_1)(e^{\gamma_2 y} - e^{\gamma_2 z})}.$$

We estimate the difference in the exponentials in the denominator as follows:

$$\begin{aligned} e^{\gamma_2 y} - e^{\gamma_2 z} &= \sum_{n=0}^{\infty} \left(\gamma_2^n \, \frac{y^n - z^n}{n!} \right) \\ &> \tfrac{\gamma_2^4}{24}(y^4 - z^4) \\ &= \tfrac{\gamma_2^4}{24}(y^2 - z^2)(y^2 + z^2) \\ &= \tfrac{\gamma_2^4}{24}(y - z)(y + z)(y^2 + z^2) > 0. \end{aligned}$$

Thus

$$\begin{aligned} G(y,z) &= \frac{k_1 + k_2(y-z) + \frac{z^2 - y^2}{\alpha} + \frac{2\mu}{\alpha^2}(y-z)}{(-\gamma_1)(e^{\gamma_2 y} - e^{\gamma_2 z})} \\ &< \frac{k_1}{(-\gamma_1)(e^{\gamma_2 y} - e^{\gamma_2 z})} \\ &\quad - \tfrac{24(k_2 + \frac{2\mu}{\alpha^2})}{(-\gamma_1)\gamma_2^4} \cdot \frac{1}{(y+z)(y^2+z^2)} \\ &\quad + \tfrac{24}{\alpha(-\gamma_1)} \cdot \frac{1}{(y^2+z^2)} \,. \end{aligned}$$

Notice the first term is strictly positive. Therefore the limit inferior will be non-negative. The other two terms converge to 0 as $(y,z) \to \infty$. This analysis implies that $F(y,z)$ is asymptotically nonnegative as $(y,z) \to \infty$ so the minimum of F is achieved in a bounded region of $\mathbb{R}_+^2$.

Due to the symmetry of F, the pair $(-y_*, -z_*)$ also minimizes F. □

Now that the lower bound given in (12) is determined, it is important to connect an optimizing μ_1^* with an admissible impulse control policy $(\tau, Y) \in \mathcal{A}_1$. The existence of two minimizing pairs (y_*, z_*) and $(-y_*, -z_*)$ allows many feasible measures μ_1 to place point masses at these two points and still achieve the lower bound. This observation leads to a solution to the restricted stochastic impulse control problem.

Theorem II.4. *Let (y_*, z_*) be a pair having positive components that minimizes F as identified in Proposition* II.3. *Consider initial positions $-y_* \le x_0 \le y_*$. Define the impulse control policy (τ^*, Y^*) as follows:*

$$\begin{cases} \tau_1^* = \inf\{t \ge 0 : X(t-) = \pm y_*\}, \\ Y_1^* = sgn(X(\tau_1^*-)) \cdot z_* - X(\tau_1^*-) \end{cases}$$

and for $k = 2, 3, 4, \ldots$, define

$$\begin{cases} \tau_k^* = \inf\{t > \tau_{k-1} : X(t-) = \pm y_*\}, \\ Y_k^* = sgn(X(\tau_k^*-)) \cdot z_* - X(\tau_k^*-). \end{cases}$$

Then (τ^*, Y^*) is an optimal impulse control pair for the restricted stochastic impulse control problem and the corresponding optimal value is

$$V_1(x_0) = g_0(x_0) + F(y_*, z_*) \cdot p_0(x_0). \tag{13}$$

Proof: The measure μ_1^* defined from (τ^*, Y^*) using (9) is concentrated on the two points $(-y_*, -z_*)$ and (y_*, z_*). Since $F(-y_*, -z_*) = F(y_*, z_*)$ and the measure $\tilde{\mu}_1^*$ having density $\frac{-Bp_0}{p_0(x_0)}$ relative to μ_1 is a probability measure, it immediately follows that the objective function value of (10) is given by (13). Therefore the impulse control pair (τ^*, Y^*) is optimal. □

C. Full Solution of Restricted Problem

Theorem II.4 proves that when the initial position x_0 is between $-y_*$ and y_*, an optimal impulse policy is to wait until X first hits either $-y_*$ or y_* and then to jump to $-z_*$ or z_*, respectively. We still need to determine an optimal policy and the optimal value for $|x_0| > y_*$. Intuition would suggest that an immediate jump from x_0 to the nearest of $\pm z_*$ followed by employing the optimal policy of Theorem II.4 might be optimal. This section uses a second linear program and its dual to verify that this intuition is correct.

Define the function $\widehat{V}$ to be

$$\widehat{V}(x) = \begin{cases} k_1 + k_2(|x| - z_*) + V_1(-z_*), & x \le -y_*, \\ V_1(x), & -y_* \le x \le y_*, \\ k_1 + k_2(x - z_*) + V_1(z_*), & x \ge y_*. \end{cases}$$

For $|x| > y_*$, the function $\widehat{V}$ is the cost associated with the process starting at initial position x, having an instantaneous jump from x to $\mathrm{sgn}(x)\, z_*$ and then using the optimal impulse control policy of Theorem II.4 thereafter. By definition, $\widehat{V}$ is symmetric about 0. The following lemma establishes the smoothness of $\widehat{V}$; the proof is fairly straightforward so is left to the reader.

Lemma II.5. $\widehat{V} \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \backslash \{\pm y_*\})$.

The function $\widehat{V}$ has sufficient smoothness to use in Dynkin's formula so (4) holds with p_0 replaced by $\widehat{V}$. Since $\widehat{V}$ is not a solution of the eigenvalue equation $Af = \alpha f$, the first term on the right-hand side of (4) no longer drops out. Using the measures μ_0 and μ_1 defined in (9), the resulting equation is

$$\int (\alpha \widehat{V} - A\widehat{V})\, d\mu_0 + \int B\widehat{V}\, d\mu_1 = \widehat{V}(x_0)$$

in which the first integral is over $\mathbb{R} - \{\pm y_*\}$. One may similarly rewrite the objective function (3) using μ_0 and μ_1, leading to a second linear program

$$\begin{cases} \text{Min.} & \displaystyle\int c_0\, d\mu_0 \quad + \int c_1\, d\mu_1 \\ \text{S.t.} & \displaystyle\int (\alpha \widehat{V} - A\widehat{V})\, d\mu_0 + \int B\widehat{V}\, d\mu_1 = \widehat{V}(x_0). \end{cases} \tag{14}$$

Instead of analyzing this linear program, we consider its dual. Since there is a single constraint, the dual linear program has a single dual variable w.

$$\begin{cases} \text{Max.} & \widehat{V}(x_0) \cdot w \\ \text{S.t.} & \left(\alpha \widehat{V}(x) - A\widehat{V}(x)\right) w \le \; c_0(x), \\ & \left(\widehat{V}(y) - \widehat{V}(z)\right) w \quad \le c_1(y,z). \end{cases} \tag{15}$$

Let $V_{dual}(x_0)$ denote the optimal value of (15), $V_{lp2}(x_0)$ denote the optimal value of (14) and recall $V_1(x_0)$ denotes the optimal value of the control problem restricted to the impulse policies in $\mathcal{A}_1$. A straightforward weak duality argument along with the same argument used to establish Theorem II.2 leads to the following bounds.

Theorem II.6. $V_{dual}(x_0) \le V_{lp2}(x_0) \le V_1(x_0)$.

We now determine the optimal solution of (15).

Theorem II.7. *The optimal value of* (15) *is* $\widehat{V}(x_0)$ *which is achieved when* $w_* = 1$.

Proof: By symmetry, it is sufficient to examine $x, y, z \ge 0$. Since $\widehat{V}(x_0)$ is the cost associated with a particular impulse control policy, it is positive so we seek as large a value for the dual variable w as possible.

We begin by examining the first family of constraints. For $-y_* < x < y_*$, $\widehat{V}(x) = g_0(x) + F(y_*, z_*)p_0(x)$ satisfies $\alpha\widehat{V}(x) - A\widehat{V}(x) = c_0(x)$ so w must be no greater than 1. Due to symmetry, it suffices to prove $\alpha\widehat{V}(x) - A\widehat{V}(x) \le c_0(x)$ for $x > y_*$. A brief analysis of the function F of (11) is needed. Proposition II.3 establishes the existence of an optimizing pair (y_*, z_*) with $0 < z_* < y_*$ but does not show uniqueness of such a pair. However the choice of y_* is critically important to verifying this first dual constraint when $x > y_*$.

Since an optimizing pair (y_*, z_*) is an interior point, the first order optimality conditions $\frac{\partial F}{\partial y}(y_*, z_*) = 0 = \frac{\partial F}{\partial z}(y_*, z_*)$ hold. Simple algebra shows these conditions are equivalent to the system

$$F(y_*, z_*) = \frac{k_2 - g_0'(y_*)}{p_0'(y_*)} = \frac{k_2 - g_0'(z_*)}{p_0'(z_*)}. \tag{16}$$

For $x > 0$, define the function h by

$$h(x) := \frac{k_2 - g_0'(x)}{p_0'(x)} = \frac{k_2 - \frac{2x}{\alpha} + \frac{2\mu}{\alpha^2}(1 - e^{\gamma_1 x})}{(-\gamma_1)\gamma_2(e^{\gamma_2 x} - e^{\gamma_1 x})}.$$

Observe that $p_0'(x) > 0$ for $x > 0$ and $k_2 - g_0'(x)$ has a unique positive root $\hat{x}$. Using the positivity of $\hat{x}$ along with its being a root of $k_2 - g_0'$, $k_2 - \frac{2\hat{x}}{\alpha} = -\frac{2\mu}{\alpha^2}(1 - e^{\gamma_1 \hat{x}}) < 0$ and hence $\hat{x} > \frac{\alpha k_2}{2}$. Therefore h is positive on the interval $(0, \hat{x})$ and negative on $(\hat{x}, \infty)$, with $h(\hat{x}) = 0$. Furthermore, $\lim_{x\to 0+} h(x) = +\infty$ and $\lim_{x\to\infty} h(x) = 0$. We now examine h':

$$\begin{aligned} h'(x) &= \frac{-g_0''(x)p_0'(x) - (k_2 - g_0'(x))p_0''(x)}{(p_0'(x))^2} \\ &= \frac{(e^{\gamma_2 x} - e^{\gamma_1 x})(\frac{2}{\alpha} + \frac{2\mu}{\alpha^2}\gamma_1 e^{\gamma_1 x})}{\gamma_1\gamma_2(e^{\gamma_1 x} - e^{\gamma_2 x})^2} \\ &\quad - \frac{(k_2 - \frac{2x}{\mu} + \frac{2\mu}{\alpha^2}(1 - e^{\gamma_1 x}))(\gamma_1 e^{\gamma_1 x} - \gamma_2 e^{\gamma_2 x})}{\gamma_1\gamma_2(e^{\gamma_1 x} - e^{\gamma_2 x})^2}. \end{aligned}$$

Note that when $x > 0$ is large, the dominant term in $h'(x)$ is $\frac{2xe^{\gamma_2 x}}{(-\gamma_1)\mu(e^{\gamma_1 x} - e^{\gamma_2 x})^2} > 0$. Thus the function h decreases from $+\infty$ at $0+$ to 0 at $\hat{x}$, becomes negative afterwards, and eventually increases to a limiting value of 0. Since $F(y_*, z_*) < 0$, there is a largest y_* such that $F(y_*, z_*) = h(y_*)$ and, with this choice of y_*, $h'(y_*) \ge 0$. As a result, $-g_0''(y_*)p_0'(y_*) - (k_2 - g_0'(y_*))p_0''(y_*) \ge 0$. We require y_* of an optimizing pair (y_*, z_*), with $0 < z_* < y_*$, to be such that $h'(y_*) \ge 0$.

We now turn to the verification of the first constraint when $x > y_*$. Using the definition of $\widehat{V}$ on $x > y_*$, we have

$$c_0(x) - (\alpha - A)\widehat{V}(x) = x^2 - \alpha k_2 x - \alpha(V_1(y_*) - k_2 y_*) - \mu k_2.$$

Hence the function $x \mapsto c_0(x) - (\alpha - A)\widehat{V}(x)$ is increasing for $x > \frac{\alpha k_2}{2}$. Since $y_* > \hat{x} > \frac{\alpha k_2}{2}$, the desired inequality $(\alpha - A)\widehat{V}(x) \le c_0(x)$ for $x \ge y_*$ will follow if we can show it holds with $x = y_*$; namely if

$$\begin{aligned} &y_*^2 - \alpha k_2 y_* - \alpha(V_1(y_*) - k_2 y_*) - \mu k_2 \\ &\quad = y_*^2 - \alpha V_1(y_*) - \mu k_2 = y_*^2 - \alpha V_1(y_*) - \mu\widehat{V}'(y_*) \ge 0. \end{aligned}$$

Observe that for $x < y_*$, $\widehat{V} = V_1 = g_0 + F(y_*, z_*)p_0$ and $\alpha V_1 - AV_1 = c_0$. In fact, extending the definition of V_1 to $(0, \infty)$, we see that $\alpha V_1(y_*) - AV_1(y_*) = c_0(y_*)$. The smoothness of $\widehat{V}$ of Lemma II.5 then establishes the third equality of

$$\begin{aligned} 0 &= y_*^2 - (\alpha - A)V_1(y_*) \\ &= y_*^2 - \alpha V_1(y_*) - \mu V_1'(y_*) + \frac{1}{2}\sigma^2 V_1''(y_*) \\ &= y_*^2 - \alpha V_1(y_*) - \mu k_2 + \frac{1}{2}\sigma^2(g_0''(y_*) + F_* p_0''(y_*)) \end{aligned} \tag{17}$$

By the choice of y_*, and the first expression for $F(y_*, z_*)$ in (16), we have

$$\begin{aligned} 0 \le h'(y_*) &= -g_0''(y_*)p_0'(y_*) - (k_2 - g_0'(y_*))p_0''(y_*) \\ &= -p_0'(y_*)[g_0''(y_*) - F(y_*, z_*)p_0''(y_*)]. \end{aligned}$$

The fact that p_0 is strictly increasing on $(0, \infty)$ so $p_0'(y_*) > 0$ implies that $g_0''(y_*) - F(y_*, z_*)p_0''(y_*) \le 0$. Using this bound in the last expression of (17) yields

$$0 \le y_*^2 - \alpha V_1(y_*) - \mu k_2,$$

showing that the first family of constraints is satisfied for $x > y_*$.

Consider now the second family of constraints with $w = 1$. There are several cases to examine. When $0 \le y \le z$, monotonicity of $\widehat{V}$ on this range shows the condition is trivially satisfied. Next, for $0 \le z \le y \le y_*$, the constraint can be rewritten as

$$V_1(y) \le k_1 + k_2(y - z) + V_1(z).$$

The right-hand expression gives the cost of an immediate jump from y to z followed by an optimal impulse control policy thereafter whereas the left-hand side gives the optimal cost. Hence this inequality is satisfied. Now consider $y_* \le z < y$ and observe that $\widehat{V}(y) - \widehat{V}(z) = k_2(y - z) < k_1 + k_2(y - z) = c_1(y, z)$. Finally, for $0 \le z < y_* < y$ and again using the definition of $\widehat{V}$, the second set of constraints in (15) is

$$[k_1 + k_2(y - z_*) + V_1(z_*)] - V_1(z) \le k_1 + k_2(y - z)$$

or, since $V_1(y_*) = k_1 + k_2(y_* - z_*) + V_1(z_*)$ is the optimal value when the initial position is y_*, equivalently

$$\begin{aligned} &k_2(y - y_*) + [k_1 + k_2(y_* - z_*) + V_1(z_*)] \\ &\quad \le k_2(y - y_*) + [k_1 + k_2(y_* - z) + V_1(z)]. \end{aligned}$$

This last inequality is true by the optimality of both the pair (y_*, z_*) and the function V_1 on $[-y_*, y_*]$ since the bracketed quantity on the right-hand side gives the cost associated with an initial impulse to z from y_* along with optimal impulse control policy starting from z. Thus the second family of constraints in (15) hold when $w = 1$. $\square$

We now have the following result.

Theorem II.8. *Let (y_*, z_*) be the optimizing pair for F having positive components. Define the impulse control policy (τ^*, Y^*) as follows;*

$$\begin{cases} \tau_1^* = \inf\{t \geq 0 : |X(t-)| \geq y_*\}, \\ Y_1^* = sgn(X(\tau_1^*-)) \cdot z_* - X(\tau_1^*-) \end{cases}$$

and for $k = 2, 3, 4, \ldots$, define

$$\begin{cases} \tau_k^* = \inf\{t > \tau_{k-1} : X(t-) = \pm y_*\}, \\ Y_k^* = sgn(X(\tau_k^*-)) \cdot z_* - X(\tau_k^*-). \end{cases}$$

Then (τ^*, Y^*) is an optimal impulse control pair for the restricted stochastic impulse control problem and the corresponding optimal value is $\widehat{V}(x_0)$.

Proof: The particular choice of (τ^*, Y^*) implies $\widehat{V}(x_0) = V_{dual}(x_0) \leq V_{lp2}(x_0) \leq V_1(x_0) \leq J(\tau^*, Y^*) = \widehat{V}(x_0)$. $\square$

III. SOLUTION FOR GENERAL ADMISSIBLE IMPULSE CONTROLS

The solution of Section II-C is restricted to those impulse control policies under which the process X remains bounded. It is necessary to show that no lower cost can be obtained by any policy which allows the process to be unbounded.

Theorem III.1. *The impulse control policy (τ^*, Y^*) of Theorem* II.8 *is optimal in the class of all admissible policies and $\widehat{V}(x_0)$ is the optimal value.*

Proof: This argument establishes that $\widehat{V}(x_0)$ is a lower bound on $J(\tau, Y; x_0)$ for every admissible impulse control policy. Theorem II.8 then gives the existence of an optimal policy whose cost equals the lower bound.

Choose $(\tau, Y) \in \mathcal{A}$ and let X be the resulting controlled process. Suppose there exists some $K > 0$ such that $\liminf_{t\to\infty} \mathbb{E}_{x_0}[e^{-\alpha t}\widehat{V}(X(t)] \geq K$. Note that

$$\begin{aligned} &\liminf_{t\to\infty} \mathbb{E}_{x_0}\left[e^{-\alpha t}\widehat{V}(X(t))\right] \\ &= \liminf_{t\to\infty} \mathbb{E}_{x_0}\left[e^{-\alpha t}\widehat{V}(X(t)) I_{\{|X(t)| \geq y_*\}}\right] \end{aligned}$$

so the linearity of $\widehat{V}$ on $\{x : |x| \geq y_*\}$ implies that $\mathbb{E}_{x_0}[|X(t)| I_{\{|X(t)|\geq y_*\}}]$ is asymptotically bounded below by $Ke^{\alpha t}$ as $t \to \infty$. Hence by Jensen's inequality for $\epsilon > 0$ and t large,

$$\mathbb{E}_{x_0}\left[X^2(t)\right] \geq \left(\mathbb{E}_{x_0}[\,|X(t)| I_{\{|X(t)|\geq y_*\}}]\right)^2 \geq K^2 e^{2\alpha t} - \epsilon.$$

Using this estimate in (3) shows $J(\tau, Y; x_0) = \infty$.

Now suppose $J(\tau, Y; x_0) < \infty$ so

$$\liminf_{t\to\infty} \mathbb{E}_{x_0}[e^{-\alpha t}\widehat{V}(X(t))] = 0.$$

Then there exists a sequence $\{t_j : j \in \mathbb{N}\}$ such that

$$\lim_{j\to\infty} \mathbb{E}_{x_0}[e^{-\alpha t_j}\widehat{V}(X(t_j))] = 0.$$

Note that $|\widehat{V}'| \leq k_2$ so $\int_0^t e^{-\alpha s}\widehat{V}'(X(s))\,dW(s)$, $t \geq 0$, is a martingale. Thus the dual constraints, in conjunction with the finiteness of the expected cost, implies that Dynkin's formula holds when $t = t_j$ for each j. Hence

$$\begin{aligned} &\widehat{V}(x_0) + \mathbb{E}_{x_0}\left[e^{-\alpha t_j}\widehat{V}(X(t_j))\right] \\ &= \mathbb{E}_{x_0}\left[\int_0^{t_j} e^{-\alpha s}[\alpha\,\widehat{V}(X(s)) - A\widehat{V}(X(s))]\,ds\right] \\ &\quad + \mathbb{E}_{x_0}\left[\sum_{k=0}^{\infty} I_{\{\tau_k \leq t_j\}} e^{-\alpha\tau_k} B\widehat{V}(X(\tau_k-), X(\tau_k))\right] \\ &\leq \mathbb{E}_{x_0}\left[\int_0^{t_j} e^{-\alpha s} c_0(X(s))\,ds\right] \\ &\quad + \mathbb{E}_{x_0}\left[\sum_{k=0}^{\infty} I_{\{\tau_k \leq t_k\}} e^{-\alpha\tau_k} c_1(X(\tau_k-), X(\tau_k))\right]. \end{aligned}$$

Letting $j \to \infty$, an application of the monotone convergence theorem on the first expectation and the convergence to 0 of $\mathbb{E}_{x_0}[e^{-\alpha t_j}\widehat{V}(X(t_j))]$ establishes that $\widehat{V}(x_0)$ is a lower bound on the expected cost $J(\tau, Y; x_0)$. $\square$

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