21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

How much better does an active price-setting newsvendor do compared to a passive one?

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***Abstract*— The classical newsvendor problem is one of the simplest decision problems under uncertainty. Given the distribution function of the random [unknown] demand for a product, and assuming that the cost and price parameters are known, a vendor decides on the number of units to be ordered/produced at the beginning of a fixed sales period; his objective, when doing so, is to maximize his expected profit when selling this batch of units on the market, see [2] and [13] for reviews. Among the many variations of the classical newsvendor problem, the extension of the model to the case where a vendor not only decides on the order size, but also chooses the selling price, is most important and has been extensively studied, see, f.i., [1], [10], [14], [15], and references therein. We call a decision maker who faces such a situation a passive price-setting newsvendor; we call the decision maker an active vendor if he dynamically adjusts prices at any instant of time during the sales period. It is a nontrivial problem to quantify the gap between the optimal expected profit of an active vendor and the expected profit of a passive one. For the following special case, we will present analytical formulas of both expressions and will derive an estimate of the difference of the two quantities: (i) the passive newsvendor faces a Poisson demand where the price elasticity of the intensity of the distribution function is constant, (ii) the active vendor deals with a similarly structured but controlled jump Markov process. In addition to the theoretical results, we will present efficient numerical schemes for computing the optimal policies, and we will discuss the results of some numerical studies.**

***Index Terms*— Optimal Control, Process Control, Stochastic Modeling and Stochastic Systems Theory**

I. INTRODUCTION

The price-setting [passive] newsvendor problem is a 2-dimensional maximization problem. If both decision variables are continuous variables and the distribution function has a density, this optimization problem can be solved by analyzing the nonlinear system of necessary optimality conditions, see e.g. [14]. Traditionally, however, it is solved as a 2-stage maximization problem as follows: **Step 1.** *Fix the price variable, and maximize the profit expression with respect to the output/inventory variable*. **Step 2.** *Maximize the resulting value function with respect to the price variable*. An alternative 2-stage approach, which will be used to analyze the case with Poisson demand, is to first solve the pricing problem, and then to solve the inventory/capacity/production problem. A perceived advantage of the former method over the latter one is the fact that the newsvendor formula for the classical problem immediately provides the solution of the inner (output) maximization problem whenever the distribution function of the random demand F is absolutely continuous. The basic ideas of both reduction approaches can be nicely illustrated if F is the exponential distribution function with mean $\mu = Ap^{-\varepsilon}$, $A, p > 0$ and $\varepsilon > 1$, i.e. $F(x) = 1 - e^{-\frac{x}{\mu}}$, $x \geq 0$. While ε specifies the constant price elasticity of demand, the parameter A equals the average number of units sold if the price is set equal to one.

A. Illustrating the traditional reduction approach

Let $\mathbb{E}^{(p)}$ denote the expectation operator for the exponential density with mean $\mu = \mu(p) = Ap^{-\varepsilon}$, and let $c > 0$ denote the unit purchasing/production cost. Let X denote the exponentially distributed random demand with intensity $1/\mu$, where $p > c$ is fixed. The optimal order quantity $q^* = q^*(p)$ of the inner maximization problem, cf. Step 1, is given by the newsvendor formula

$$1 - F(q^*) = \frac{c}{p}; \tag{1}$$

for each p fixed, the corresponding expected profit value is given by

$$\begin{aligned} V(p, q^*(p)) &:= \max_{q \geq 0} \left\{ p\mathbb{E}^{(p)}[q \wedge X] - c \cdot q \right\} \\ &= p\mathbb{E}^{(p)}[q^* \wedge X] - c \cdot q^* \\ &= p \int_0^{q^*} \frac{x}{\mu} e^{-\frac{x}{\mu}} dx. \end{aligned} \tag{2}$$

In the exponential case, formulas (1) and (2) yield the following expressions for the optimal order quantity q^*, $q^*(p) = \mu(p) \log(p/c)$, and for the corresponding expected profit as a function of p,

$$V(p, q^*(p)) = \mu(p) \left[p - c(\log(p/c) + 1) \right], \tag{3}$$

see [7] for expressions which look different but are equivalent.

Simple calculations show that the necessary optimality condition of the outer maximization problem, cf. Step 2, i.e. differentiate (3) with respect to p and set the derivative equal to zero, is equivalent to the equation

$$\frac{\varepsilon}{\varepsilon - 1} \log\left(\frac{p}{c}\right) = \frac{p}{c} - 1. \tag{4}$$

Identity (4) implicitly determines the optimal price $p^* > c$. It follows from (4) that the ratio of p^* and c equals a constant $\kappa > 1$. As a function of ε, this ratio value $\kappa(\varepsilon)$ is monotone decreasing. Hence, the optimal retail price p^* tends to c if the newsvendor faces very price sensitive customers, i.e., let $\varepsilon \to \infty$.

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ISBN: 978-90-367-6321-9

Combining the various expressions, we obtain the following formulas for the optimal decisions variables and the optimal value of a passive [price-setting] newsvendor with exponentially distributed demand; the fundamental assumption is that the mean of the distribution has constant price elasticity:

$$p^* = \kappa \cdot c \tag{5}$$

$$q^* = \log(\kappa) \cdot \mu(p^*) \tag{6}$$

$$V^* = V(p^*, q^*) = \frac{1}{\varepsilon}(p^* - c)\mu(p^*); \tag{7}$$

see [1] for general expressions of p^* and q^*. For this very special example, formulas (5)-(7) justify the following management recommendations:

(i) Use a simple markup rule to specify the optimal price.
(ii) Whenever the logarithm of the markup factor is bigger than one, choose the optimal batch size/capacity to be bigger than the expected demand; otherwise, choose the batch size to be smaller than the expected demand.
(iii) The optimal expected profit equals $1/\varepsilon$ times the product of the number of expected buyers and the profit per unit sold.

B. Illustrating the alternative reduction approach

This time, the "inner" maximization will be with respect to the price variable, while the outer maximization will be with respect to quantity. Let q be given; maximizing the expected revenue $p \cdot \mathbb{E}^{(p)}[q \wedge X]$ is not restricted by cost considerations. In the case of an exponential distribution, integration by parts yields the following formula for the expected revenue:

$$p\mathbb{E}^{(p)}[q \wedge X] = p\mu(p)F(q; \mu(p)), \tag{8}$$

where $F(q;\mu) = 1 - e^{-q/\mu}$. We denote the density of F by $f(q;\mu) = 1/\mu e^{-q/\mu}$, $q \geq 0$. The necessary optimality condition, when maximizing (8) with respect to p, is equivalent to the identity

$$e^{q/\mu(p)} - 1 = \frac{\varepsilon}{\varepsilon - 1}\frac{q}{\mu(p)}. \tag{9}$$

Analyzing (9) we find that, (i) for any $q > 0$ there is a unique $p > 0$, not necessarily bigger than c, which satisfies (9), (ii) there is a unique positive value ϱ such that

$$q/\mu(p) = \varrho, \tag{10}$$

i.e. $\varrho > 0$ and, see (9),

$$e^{\varrho} - 1 = \frac{\varepsilon}{\varepsilon - 1}\varrho. \tag{11}$$

It can be easily seen, cf. (4) and (9), that $e^{\varrho} = \kappa$. The identity (3) implicitly defines a function $q \mapsto \hat{p}(q)$, which associates with a quantity q its best associated retail price $\hat{p}$.

The second step of the alternative reduction approach requires maximizing the expected profit

$$\hat{p}(q)\mu(\hat{p}(q))F(q;\mu(\hat{p}(q))) - cq \tag{12}$$

with respect to q. Using (10) and the relation $\mu(p) = Ap^{-\varepsilon}$, simple algebra shows that this objective function is identical to

$$Const \cdot q^{1-\frac{1}{\varepsilon}} - c \cdot q, \tag{13}$$

where $Const := (A\varrho)^{1/\varepsilon} \cdot (1 - e^{-\varrho})/\varrho$. Analyzing the necessary optimality condition of the outer maximization problem, the optimal order quantity q^* can be characterized as follows:

$$q^* = \frac{A\varrho e^{-\varepsilon\varrho}}{c^{\varepsilon}}; \tag{14}$$

there are other, equivalent formulas for q^*. Since $\kappa = e^{\varrho}$, we also obtain the formulas $p^* = \hat{p}(q^*) = ce^{\varrho} = \kappa c$, $q^* = \varrho\mu(p^*)$, and $V^* = p^* E^{(p^*)}[q^* \wedge X] - c \cdot q^*$ is equal to the right hand side of (7); these three formulas are consistent with the expressions (5)-(7).

Remark 1.1: Since (4) is independent of A, the optimal price p^* does not depend on A. However, the optimal order quantity and the expected profit depend on the number of expected shoppers.

Remark 1.2: Throughout the paper, we do not spell out details concerning second order optimality conditions. Verifying these conditions is usually straightforward; sometimes, however, the calculations are somewhat tedious.

This short paper is organized as follows. In Section II we prove several analytical results relevant for the study of passive price-setting newsvendor problems with Poisson demand and iso-elastic mean. In particular, we show the sequence of expected profits as a function of the initial inventory level to be unimodal. In Section III, we introduce the active newsvendor problem and discuss the relationship of both newsvendor problems. In Section IV we prove a comparison theorem and describe different routines for numerically solving these problems. We also report on some numerical studies. We conclude the paper with Section V where we also position the paper to some references.

II. The passive newsvendor with Poisson demand

In this section we shall analyze the case of a Poisson demand distribution $F(n;\lambda)$, $n = 0, 1, 2, \ldots$, where $\lambda = \lambda(p) := Ap^{-\varepsilon}$, $A > 0$, $\varepsilon > 1$. Whenever the demand distribution is discrete, and an arbitrary price p is given, the newsvendor formula is not an equation but an inequality: "order up to $n^*(p)$ such that all (backward) increments of the expected profit function are positive". In the Poisson case, in order to avoid dealing with an inequality characterization right away, we adopt the alternative reduction approach. Moreover, to stress the difference between the discrete case and the continuous one, the quantity variable will be denoted by n instead of q. For any $n \in \mathbb{N}$, we first maximize the expected revenue with respect to the continuous decision variable p, and then maximize the sequence of (optimal) expected profits with respect to the order quantity n.

As in Section I.A, we let $p > 0$, $A > 0$ and $\varepsilon > 1$, and assume $\lambda = \lambda(p) := Ap^{-\varepsilon}$. We denote the Poisson distribution function with parameter λ by $F(n;\lambda) = e^{-\lambda}\sum_{k=0}^{n}\frac{\lambda^k}{k!}$, $n = 0, 1, 2, \ldots$ In the sequel, X will always denote a $\mathbb{N}_0$-valued random variable with distribution function $F = F(\cdot;\lambda)$; the exact value of λ will be clear from the context. For any (fixed) $n \in \mathbb{N}_0$, $p > 0$, the expected revenue $p\mathbb{E}^{(p)}[n \wedge X]$ can be expressed in many different but equivalent ways. The following expression is the analogue of (8), $\lambda = \lambda(p)$,

$$p\mathbb{E}^{(p)}[n \wedge X] = p\lambda F(n-1;\lambda) + pn(1 - F(n;\lambda)). \quad (15)$$

The proof is an elementary exercise and will be omitted.

The starting point of our analysis of the passive newsvendor problem with Poisson demand and mean λ, which has constant price elasticity, is the necessary optimality condition which p has to satisfy if (15) is maximized for a given n.

Lemma 2.1: For any $n \in \mathbb{N}$, the unique maximizer p_n of (15) satisfies, $\lambda_n := \lambda(p_n)$,

$$1 - F(n;\lambda_n) = (\varepsilon - 1)\frac{\lambda_n}{n} F(n-1;\lambda_n). \quad (16)$$

Proof. Since $p = (A/\lambda)^{1/\varepsilon}$, maximizing (15) with respect to p is equivalent to maximizing (15) with respect to λ; replace the variable p by $(A/\lambda)^{1/\varepsilon}$. The derivative of $F(n;\lambda)$ with respect to λ, after rearranging terms, equals

$$-e^{-\lambda}\lambda^n/n!.$$

Next, taking the derivative of all terms of (15), simple algebra shows that the necessary optimality condition (in λ) is equivalent to (16). It remains to show that (16) has a unique positive solution value λ_n; the corresponding price value is then given by $p_n := (A/\lambda_n)^{1/\varepsilon}$. Multiplying both sides of (16) by $\exp(\lambda_n)$, we obtain the following equation that λ_n needs to satisfy, $\lambda > 0$,

$$\sum_{k=n+1}^{\infty} \frac{\lambda^k}{k!} = (\varepsilon - 1)\frac{\lambda}{n}\sum_{k=0}^{n-1}\frac{\lambda^k}{k!}. \quad (17)$$

Dividing both sides of (17) by $\lambda^{n+1}/(n+1)!$, we obtain

$$\begin{aligned} &1 + \frac{\lambda}{(n+2)} + \frac{\lambda^2}{(n+2)(n+3)} + \cdots \quad (18) \\ &= \quad (\varepsilon-1)(n+1)\left(\frac{1}{\lambda} + \frac{(n-1)}{\lambda^2} + \cdots + \frac{(n-1)!}{\lambda^n}\right). \end{aligned}$$

Let $\mathcal{L}_n(\lambda)$, $\mathcal{R}_n(\lambda)$ resp., denote the left, right resp., hand side of (18). Obviously, for any $n \in \mathbb{N}$, $\mathcal{L}_n$ is a strictly monotone increasing function of λ while $\mathcal{R}_n$ is strictly monotone decreasing. Since $\mathcal{R}_n(\lambda)$ lies above $\mathcal{L}_n(\lambda)$ if λ is close to zero, and $\mathcal{R}_n(\lambda)$ stays below $\mathcal{L}_n(\lambda)$ if λ is large, there is a unique solution λ_n. ■

There are other properties of the sequence $(\lambda_n)_n$ which can be deduced from (18). Without proof, we state two of these properties; the property of the sequence $(p_n)_n$ is obvious

Lemma 2.2: Let λ_n be the unique solution of (18).

(i) The sequence $(\lambda_n)_n$ is monotone increasing; the sequence of optimal prices $(p_n)_n$ is monotone decreasing.

(ii) The function $\lambda_n(\varepsilon)$ is monotone increasing in ε, $\varepsilon > 1$, $n \in \mathbb{N}$ fixed.

In constrast to the simple case of the exponential distribution, cf. Section I.B, in the discrete case the ratios λ_n/n vary with n but stabilize around one.

Lemma 2.3: $\lim_{n\to\infty} \lambda_n/n = 1$.

Proof. Let $Z_n := \sum_{k=0}^{n-1} \frac{\lambda_n^k}{k!}$, and consider (17). Multiplying both sides of (17) by e^{λ_n}, we get

$$e^{\lambda_n} - \sum_{k=0}^{n}\frac{\lambda_n^k}{k!} = e^{\lambda_n} - \frac{\lambda_n^n}{n!} - Z_n = \frac{\lambda_n}{n}(\varepsilon-1)Z_n.$$

Hence,

$$e^{\lambda_n} = \frac{\lambda_n^n}{n!} + \left(1 + (\varepsilon-1)\frac{\lambda_n}{n}\right)Z_n. \quad (19)$$

By definition and elementary but lengthy algebraic manipulations of (18) and (19), we derive the following upper bound for Z_n,

$$Z_n < \frac{\varepsilon}{\varepsilon-1}\frac{\lambda_n^n}{(n-1)!}. \quad (20)$$

Factoring out the term $\frac{\lambda_n^n}{n!}$ from the right hand side of (19) and taking the n-th root of the resulting expression, we get

$$e^{\frac{\lambda_n}{n}} = \frac{n}{(n!)^{1/n}}\frac{\lambda_n}{n}\left(1 + \left(1 + (\varepsilon-1)\frac{\lambda_n}{n}\right)\frac{n!}{\lambda_n^n}Z_n\right)^{1/n}.$$

It is well known that the first factor of this product converges to e from below. Using (20), we obtain

$$\begin{aligned} e^{\frac{\lambda_n}{n}} &< e\cdot\frac{\lambda_n}{n}\left(1 + \left(1 + (\varepsilon-1)\frac{\lambda_n}{n}\right)\frac{n!}{\lambda_n^n}Z_n\right)^{1/n} \\ &< e\frac{\lambda_n}{n}\left(1 + \left(1 + (\varepsilon-1)\frac{\lambda_n}{n}\right)\frac{n!}{\lambda_n^n}\frac{\varepsilon}{\varepsilon-1}\frac{\lambda_n^n}{(n-1)!}\right)^{1/n} \\ &= e\cdot\frac{\lambda_n}{n}\left(1 + \frac{\varepsilon}{\varepsilon-1}(n + (\varepsilon-1)\lambda_n)\right)^{1/n}. \end{aligned}$$

Multiplying by e^{-1} and pulling out the factor n inside the parenthesis, we get

$$e^{\frac{\lambda_n}{n}-1} < \frac{\lambda_n}{n}\left(1 + \frac{\varepsilon}{\varepsilon-1}n(1 + (\varepsilon-1)\frac{\lambda_n}{n})\right)^{1/n}. \quad (21)$$

It follows from (21) that the sequence $(\lambda_n/n)_n$ is bounded. Next, we shall verify that the sequence $(\lambda_n/n)_n$ has a limit which equals one. To this end, take any sequence $(n_k)_k$ such that λ_{n_k}/n_k converges to a number ξ if n_k tends to infinity. Since inequality (21) also holds for n_k, we get

$$e^{\xi-1} \le \xi\left(\lim_{n_k\to\infty}(1 + n_k\cdot\varphi_{n_k})\right)^{1/n_k}, \quad (22)$$

where $\varphi_{n_k} \to \frac{\varepsilon}{\varepsilon-1}(1 + (\varepsilon-1)\xi)$. The second factor of the product expression (22) converges to 1, and the resulting inequality implies the assertion. ■

To solve the outer optimization problem in the Poisson case (maximizing with respect to $n \in \mathbb{N}_0$), and looking ahead to the comparison of a passive newsvendor with an active one, we express the expected profit $G_n := p_n\mathbb{E}^{(p_n)}[n\wedge X] - c\cdot n$, in two different ways. Let W_n denote the first term of the difference expression for G_n, and recall that p_n denotes the best price of the revenue maximization problem when n is given, see (15).

Theorem 2.1: For any $n \in \mathbb{N}$, let p_n be defined as above, see proof of Lemma 2.1. Let

$$\mathfrak{z}_n := \varepsilon\lambda_n^{\frac{\varepsilon-1}{\varepsilon}}\frac{1 - e^{-\lambda_n}\frac{\lambda_n^n}{n!}}{1 + (\varepsilon-1)\frac{\lambda_n}{n}}. \quad (23)$$

Then, $G_n = \varepsilon p_n \lambda_n F(n-1, \lambda_n) - cn = A^{1/\varepsilon} \mathfrak{z}_n - cn$.

Proof. By definition, see also the explanation prior to Lemma 2.1, and by (17), we have

$$\begin{aligned} G_n &= \lambda_n p_n F(n-1;\lambda_n) + n p_n (1 - F(n;\lambda_n)) - cn \\ &= \lambda_n p_n F(n-1;\lambda_n) + n p_n (\varepsilon - 1)\frac{\lambda_n}{n} F(n-1;\lambda_n) \\ &\quad - cn \\ &= \varepsilon \lambda_n p_n F(n-1;\lambda_n) - cn. \end{aligned}$$

Using the relation $p_n = (A/\lambda_n)^{1/\varepsilon}$, and exploiting (19) when rewriting (18), we obtain the second expression for G_n, as well as the formula

$$W_n = \mathfrak{z}_n A^{1/\varepsilon}. \tag{24}$$

■

Corollary 2.1: $\lim_{n\to\infty} \frac{\lambda_n^n}{n!} e^{-\lambda_n} = 0$, and $\lim_{n\to\infty} F(n;\lambda_n) = \lim_{n\to\infty} F(n-1;\lambda_n) = \frac{1}{\varepsilon}$.

From a numerical point of view, the next result is most important. It ensures that a "Golden-Section-Search" can be used to find (n^*, p^*), the optimal order quantity n^* and the optimal price p^* of the 2-dimensional (passive) newsvendor problem.

Theorem 2.2: The sequence $(G_n)_n$ is unimodal.

Proof. We only give a sketch of the proof since the details of the induction arguments are involved. Actually, our proof is one by picture(s), see Figures 1-2.

For every $n \in \mathbb{N}$, let p_n be the solution of the pricing problem, and let $\hat{n}(p_n)$ denote the optimal order quantity of the capacity problem of a classical newsvendor, i.e. $\hat{n}(n) := \hat{n}(p_n) = \min\{k \in \mathbb{N}_0 \mid F(k;p_n) \geq (p_n - c)/p_n\}$.

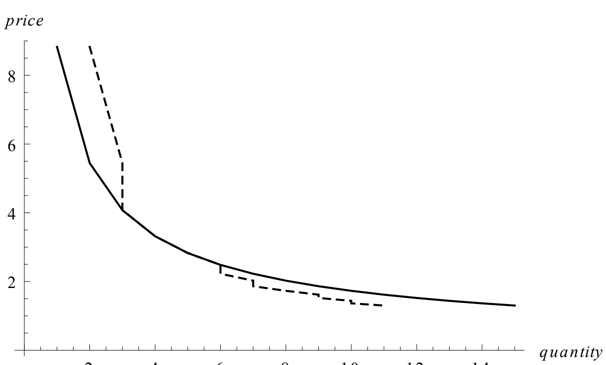

Fig. 1. The graphs of $\{(n, p_n)\}$, solid line, and $\{(\hat{n}(p_n), p_n)\}$, dashed line; the parameters are: $A = 20$, $\varepsilon = 1.5$ and $c = 1$.

We assume $p_1 > c$; otherwise, the problem is trivial. For "small" values of n, p_n will be large, cf. Table I, and $\hat{n}(p_n)$ will be bigger than n. If n increases, then p_n will (strictly) decrease and the difference $(\hat{n}(p_n) - n)$ will become zero for some values of n. We denote the first, last resp., n for which this happens by n_u, n_0 resp.; we set $\mathfrak{K} = \{n_u, n_u+1, \ldots, n_0\} = \{n \in \mathbb{N}_0 \mid \hat{n}(p_n) - n = 0\}$. It follows from properties of $\hat{n}(p_n)$, p_n, G_n and $G_{\hat{n}(p_n)}$ that n^*, the optimal order quantity of the passive price-setting newsvendor, belongs to $\mathfrak{K}$, see Figure 1, Figure 2

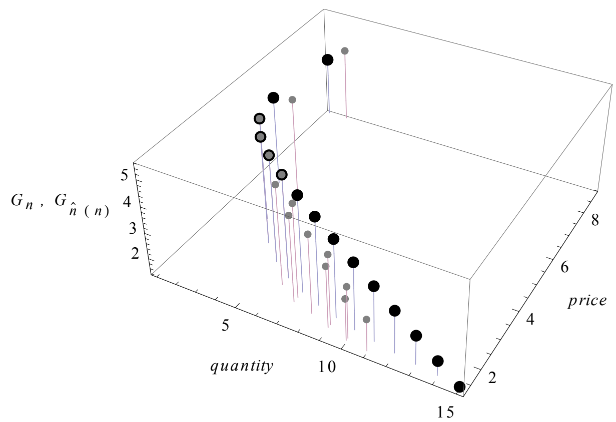

Fig. 2. The values G_n and $G_{\hat{n}(p_n)}$ defined at the points $\{(p_n, n)\}$ and $\{(p_n, \hat{n}(p_n))\}$; the parameters are: $A = 20$, $\varepsilon = 1.5$ and $c = 1$. The black dots are the values of G_n and the gray ones are the values of $G_{\hat{n}(p_n)}$.

TABLE I

NUMERICAL VALUES OF p_n, $\hat{n}(p_n)$, G_n AND $G_{\hat{n}(p_n)}$ AS FUNCTIONS OF GIVEN CAPACITY n, $n = 1, 2, \ldots, 15$; THE PARAMETERS ARE: $A = 20$, $\varepsilon = 1.5$ AND $c = 1$.

n	p_n	$\hat{n}(p_n)$	G_n	$G_{\hat{n}(p_n)}$
1	8.8265	2	3.70973	4.27963
2	5.44582	3	4.85781	5.00078
3	4.07648	3	5.34901	5.34901
4	3.31754	4	5.52283	5.52283
5	2.82834	5	5.50535	5.50535
6	2.48353	6	5.35825	5.35825
7	2.22567	6	5.11672	5.23213
8	2.02454	7	4.80285	5.00766
9	1.86264	7	4.43154	4.72284
10	1.7291	8	4.01332	4.45196
11	1.61678	9	3.55597	4.11582
12	1.52079	9	3.0654	3.78097
13	1.43766	10	2.54621	3.41752
14	1.36486	10	2.00207	3.0422
15	1.30049	11	1.43594	2.65449

and also Table 1. The convex curve (solid line in Fig. 1) is determined by the points (n, p_n), while the dashed line connects the points $(p_n, \hat{n}(p_n))$; for the special example, $\mathfrak{K} = \{3, 4, 5, 6\}$. The interplay of both reduction approaches implies that $G_n = G_{\hat{n}(p_n)}$ if $n \in \mathfrak{K}$, and that both sequences $(G_n)_n$ as well as $(G_{\hat{n}(p_n)})_n$ are unimodal. ■

Figure 3 illustrates the fact that both sequences are unimodal, and that $(G_{\hat{n}(p_n)})_n$ dominates $(G_n)_n$.

III. THE ACTIVE NEWSVENDOR PROBLEM

The active newsvendor problem was introduced en passant in [5]. It consists of two [staged] optimization problems, a particular dynamic pricing problem with no cost considerations, and a capacity problem which builds on the solution of the pricing problem. The authors of [5] called the latter a newsvendor-like problem. For us, the active newsvendor problem consists of the dynamic pricing problem with constant price elasticity which has been analyzed and solved by McAfee and te Velde [8], see also [5], and the capacity problem. The dynamic pricing problem is a generalization of

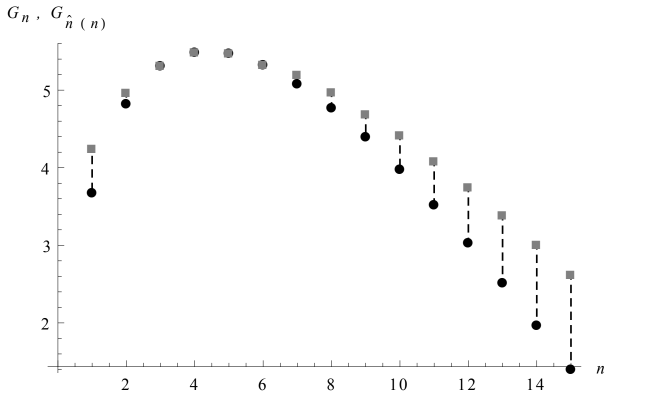

Fig. 3. The values G_n (black dots) and $G_{\hat{n}(p_n)}$ (grey squares) as functions of n, $n = 1, 2, \ldots, 15$; the parameters are: $A = 20$, $\varepsilon = 1.5$ and $c = 1$.

the inner (static pricing) maximization problem of a passive newsvendor, cf. Section II and Theorem 2.1. Instead of a fixed Poisson distribution with mean $\lambda = Ap^{-\varepsilon}$, the active decision maker assumes customers, who are willing to pay the price p for one unit of the product at time t, $0 \le t < T$, to arrive at the rate $\Lambda(p,t) := a(t)p^{-\varepsilon}$, $a(t)$ a positive function defined on the sales period $[0,T]$, $T < \infty$.

For any t, let $A(t) := \int_t^T a(s)ds$; we set $A := A(0)$. If $a(t)$ is a constant, i.e. $a(t) \equiv a$, then $A = aT$; it is obvious that the length of the sales period is an important parameter for an active vendor. In contrast to a passive vendor, an active newsvendor will set a new price at any instance of time t, $0 \le t < T$. If, however, the pricing policy dictates the price to be the same for every t, $p(t) \equiv \bar{p}$, $\bar{p} > 0$, then the active vendor becomes a passive one, and the number of customers willing to buy at price $\bar{p}$ throughout the period T is Poisson distributed with mean $\lambda(\bar{p}) = A\bar{p}^{-\varepsilon}$. For general Markov feedback policies, however, the number of buyers is a jump Markov process with (controlled) Poisson rates, and distributional characteristics which are no longer described by a simple Poisson distribution, cf. [6] for details.

Let $V(n,t)$ represent the continuation value (independent of any prior sales) to the vendor of having $n \in \mathbb{N}_0$ items to sell at time t. The following Theorem, see [8], characterizes the value function $V(n,t)$ and the optimal Markov policy $p^*(n,t)$; see [5] for a generalization. Let $(\beta_n)_{n=0,1,\ldots}$ be the strictly increasing sequence of nonnegative numbers which satisfy the recursion $\beta_0 = 0$, $\beta_n > \beta_{n-1}$ and β_n solves $\beta_n = ((\varepsilon-1)/\varepsilon)^{\varepsilon-1}(\beta_n - \beta_{n-1})^{-(\varepsilon-1)}$, see [8] and [5] for more details.

Theorem 3.1: Let n be any natural number. The expected *revenue* V_n of the pricing part of the active newsvendor problem equals

$$V_n := V(n,0) = \beta_n A^{1/\varepsilon}, \tag{25}$$

and the optimal pricing policy (in feedback form) is given by

$$p^*(t,n) = \beta_n^{-1/(\varepsilon-1)} A(t)^{1/\varepsilon}.$$

Recall, the active newsvendor problem comprises the (optimal) dynamic pricing problem described above, and the following economic order quantity (EOQ) problem, $c > 0$,

$$\bar{H} := \max_{n \in \mathbb{N}_0} \{V_n - cn\} =: \max_{n \in \mathbb{N}_0} \{H_n\}. \tag{26}$$

The sequence $(\beta_n)_n$ has several most interesting properties, see [5] and [8] for more details. In particular, the strictly monotone increasing sequence is concave, and the values β_n asymptotically behave like $n^{(\varepsilon-1)/\varepsilon}$; to be precise,

$$\frac{\beta_n}{n^{1-1/\varepsilon}} \nearrow 1, \qquad \text{if } n \to \infty. \tag{27}$$

The properties of $(\beta_n)_n$ imply that the objective function of (26) is strictly concave and unimodal. Thus, there is a unique ordering quantity $\bar{n}$ which can be characterized in different ways. For instance, $\bar{n}$ equals the smallest integer value such that

$$H_{\bar{n}} - H_{\bar{n}-1} \ge 0, \quad \text{but} \quad H_{\bar{n}+1} - H_{\bar{n}} < 0.$$

Taking Theorem 2.1 and the fact that β_n satisfies a first order (nonlinear) difference equation into account, simple algebra yields the following explicit characterization of $\bar{n}$.

Theorem 3.2:

$$\bar{n} = \max_{n \in \mathbb{N}_0} \left\{ n \;\middle|\; \beta_n \le \left(\frac{\varepsilon-1}{\varepsilon c}\right)^{\varepsilon-1} A^{\frac{\varepsilon-1}{\varepsilon}} \right\}, \tag{28}$$

$$\bar{H} = \beta_{\bar{n}} A^{1/\varepsilon} - c\bar{n}. \tag{29}$$

In the homogeneous case, i.e. $a(t) \equiv a$, and if $\bar{n}$ is large,

$$\bar{n} \approx \left(\frac{\varepsilon-1}{\varepsilon c}\right)^{\varepsilon} A, \quad \text{where } A = aT, \tag{30}$$

$$\bar{H} \le \left(\frac{\varepsilon-1}{\varepsilon c}\right)^{\varepsilon-1} A - c\bar{n} \approx \frac{A}{\varepsilon}\left(\frac{\varepsilon-1}{\varepsilon c}\right)^{\varepsilon-1}. \tag{31}$$

The characterization of $\bar{n}$ as being a threshold type is most useful since the sequence $(\beta_n)_n$ is monotone increasing and the values β_n are recursively computed.

Theorem 3.2 shows that the optimal order size $\bar{n}$ and the optimal value $\bar{H}$ of an active vendor increase with the length of the sales period T; both values also increase with the arrival rate a, but decrease with c.

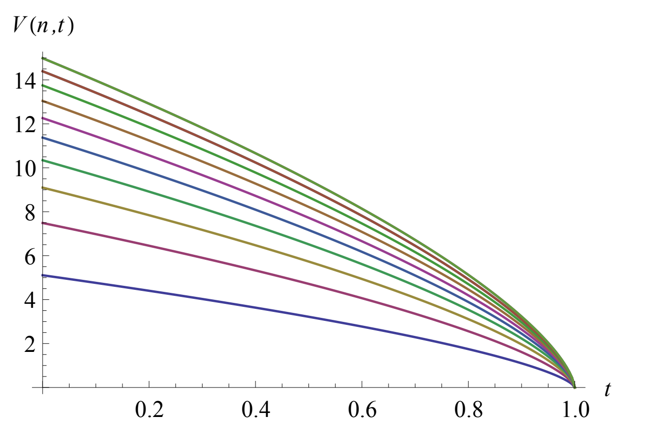

Fig. 4. The value function (optimal expected reward) $V(n,t)$, see (25), as a function of t, $0 \le t \le T$; the parameters are: $T = 1$, $a = 2$, i.e. $A = 20$, $\varepsilon = 1.5$. The top curve shows $V(10,t)$ and the bottom one $V(1,t)$.

For different values of n, Figure 4 and Figure 5 display typical graphs of the value functions $V(n,t)$ and $p(n,t)$, $0 \le t \le T$. The initial values $V_n = V(n,0)$ are, up to the common factor of proportionality $A^{1/\varepsilon}$, equal to β_n. The graphs show that – naturally – the values V_n are increasing (in n) and, as should be expected, the marginal profits $V_{n+1} - V_n$ are decreasing with the number of items to be sold.

The values $p(n,0)$ differ from the values p_n, but the difference is "small" if n is "large".

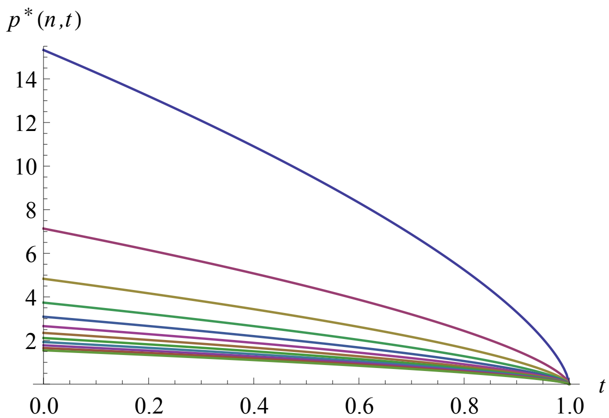

Fig. 5. The optimal (dynamic) pricing policy $p^*(n,t)$ as functions of t, $0 \le t \le T$, cf. Section III, for various $n \in \mathbb{N}$; the parameters are: $T = 1$, $a = 20$, i.e. $A = 20$, $\varepsilon = 1.5$. The top graph represents $p^*(1,t)$ while the graph at the bottom shows $p^*(10,t)$.

IV. An active newsvendor versus a passive vendor

For Poisson rates $\Lambda(t,n)$, see above, an active newsvendor who restricts himself to constant price strategies is identical with a passive newsvendor. Therefore, the following inequality holds for any $n \in \mathbb{N}_0$,

$$W_n \le V_n; \tag{32}$$

furthermore,

$$\bar{G} = \max_{n \in \mathbb{N}_0} \{G_n\} \le \bar{H}. \tag{33}$$

The fact that the numbers $\mathfrak{z}_n$, s. (23), are proportional to the optimal values of particular (static) pricing problems yields the following nontrivial result.

Proposition 4.1: The sequence $(\mathfrak{z}_n)_n$ is monotone increasing

Proof. Consider the revenue maximization problems with endowments n and $(n+1)$. Since a price setting vendor with $(n+1)$ items to sell can simply dump the $(n+1)$-th item and set the retail price equal to p_n, his (price optimal) expected revenue W_{n+1} exceeds W_n. Using (24), the property that the sequence $(\mathfrak{z}_n)_n$ is monotone increasing follows. ∎

The next result compares the values W_n of a passive vendor with the values V_n of an active one, see formulas (24) and (25).

Proposition 4.2: $\forall\, h > 0 \;\exists\; N_h \;\forall\, n \ge N_h$,

$$(1-h) n^{\frac{\varepsilon-1}{\varepsilon}} \;\le\; \mathfrak{z}_n \;\le\; \beta_n \;\le\; n^{\frac{\varepsilon-1}{\varepsilon}}. \tag{34}$$

Proof. Proposition 4.1 and formula (27) justify the second inequality and the third one. To see that the first inequality holds, recall Stirling's formula, e.g. [3],

$$\sqrt{2\pi n}\frac{n^n e^{-n}}{n!} \to 1, \quad \text{if} \quad n \to \infty.$$

Elementary calculus implies that for every $n \in \mathbb{N}$ fixed, and $\lambda > 0$,

$$\frac{\lambda^n e^{-\lambda}}{n!} \le \frac{n^n e^{-n}}{n!}.$$

Thus, for any $n \in \mathbb{N}$, see (23),

$$1 - \frac{\lambda_n^n}{n!} e^{-\lambda_n} \ge 1 - \frac{n^n}{n!} e^{-n}.$$

Moreover, see again (23),

$$\max_{\xi \ge 0} \left\{ \frac{\varepsilon \xi^{\frac{\varepsilon-1}{\varepsilon}}}{1 + (\varepsilon - 1)\xi} \right\} = 1.$$

Hence, if $h > 0$ is given, choose N_h large enough such that for any $n \ge N_h$ the ratio λ_n / n is close to one, and the inequalities

$$\frac{\varepsilon(\lambda_n/n)^{\frac{\varepsilon-1}{\varepsilon}}}{1 + (\varepsilon-1)\lambda_n/n} \ge \sqrt{1-h}, \quad 1 - \frac{n^n e^{-n}}{n!} \ge \sqrt{1-h}$$

hold. Finally, if n exceeds N_h, then the first inequality of (34) follows by construction from (23) and Stirling's formula. ∎

The theoretical results of Section II and III, suggest the following recommendations when numerically solving active or passive newsvendor problems:

(i) If $\left(\frac{\varepsilon-1}{\varepsilon c}\right)^{\varepsilon} A$ is "small" (a few hundred), then the optimal order quantity n^* will be relatively "small", and n^* is quickly determined by computing the differences $G_n - G_{n-1}$, $n \ge 1$, and by testing the positivity of these values.

(ii) If $\left(\frac{\varepsilon-1}{\varepsilon c}\right)^{\varepsilon} A$ is "large", then n^* will be large. Choose $\tilde{p} := \left(\frac{\varepsilon}{\varepsilon-1} c\right)$ as a first estimate of p^*, and $\tilde{n} := A\tilde{p}^{-\varepsilon}$ as a first estimate of n^*. Take a guess, f.i. $\tilde{\mathfrak{K}} := \{0.95\tilde{n}, \ldots, 1.05\tilde{n}\} \in \mathbb{N}$, and optimize G_n on $\tilde{\mathfrak{K}}$, either by pointwise evaluations and simple comparisons, or by a Golden-Section-Search. If n^* lies inside $\tilde{\mathfrak{K}}$, stop. If n^* is a boundary point, then enlarge the set appropriately.

(iii) For large values of $\tilde{n}$ replace β_n by $n^{(\varepsilon-1)/\varepsilon}$ (from above), see Theorem 3.2, and by $\mathfrak{z}_n$ (from below). While β_n is recursively computed, the values $\mathfrak{z}_n$ can be individually computed by solving the nonlinear equation (16). To this end, in-built commands for inverse distribution functions, cf. Matlab, Mathematica, etc., are most helpful.

These recommendations can be justified as follows: If n is large, then $\lambda_n / n \approx 1$, and $1 - F(n; \lambda_n) \approx 1/\varepsilon$, see Lemma 2.3 and Corollary 2.1. Moreover, $\lambda_n = A p_n^{-\varepsilon}$, and $p_n = (A/\lambda_n)^{1/\varepsilon} = \left(A \frac{n}{\lambda_n} n\right)^{1/\varepsilon} \approx (An)^{1/\varepsilon}$, see proof of Lemma 2.1 and Lemma 2.3; furthermore, $p_{\tilde{n}} \approx p^* \approx \frac{\varepsilon}{\varepsilon-1} c$, see Sections II and III.

The following numerical examples illustrate how these recommendations can be used.

TABLE II

NUMERICAL VALUES OF $\mathfrak{z}_n$, β_n AND $n^{(\epsilon-1)/\epsilon}$ FOR $\varepsilon = 1.5$.

n	$\mathfrak{z}_n$	β_n	$n^{(\epsilon-1)/\epsilon}$
1	0.639208	0.693361	1.
2	0.930748	1.01617	1.25992
3	1.13313	1.23479	1.44225
100	4.47148	4.6043	4.64159
200	5.69681	5.82234	5.84804
300	6.55313	6.67373	6.69433
400	7.23355	7.35047	7.36806
500	7.80746	7.92146	7.93701
600	8.3087	8.42027	8.43433
700	8.75663	8.86614	8.87904
800	9.16348	9.27121	9.28318
900	9.53755	9.64369	9.65489
1000	9.88471	9.98944	10.

TABLE III

RUN TIMES (IN SECONDS) OF THE MATHEMATICA ROUTINES R1, R2 AND R3 DESCRIBED ABOVE. PARAMETER VALUES: $\varepsilon = 1.5$ AND $c = 1$.

	R1	R2	R3
$A = 20$	0.017	0.011	0.023
$A = 20,000$	550.2	322.5	12.9

While Proposition 4.2 guarantees $\lim_{n\to\infty} \beta_n/n^{\frac{\varepsilon-1}{\varepsilon}} = \lim_{n\to\infty} \mathfrak{z}_n/n^{\frac{\varepsilon-1}{\varepsilon}} = 1$, Table II shows that - usually - convergence is slow. Thus, recommendation (iii) needs to be taken with a grain of salt. Table III is a synopsis of numerical studies related to recommendation (ii). For the parameter values $\varepsilon = 1.5$, $c = 1$ and $A = 20$ as well as $A = 20,000$ we report the run times of simple (homemade) Mathematica subroutines of the following procedures: (R1) compute (backward) differences of the sequence $(G_n)_n$ and stop when the signum of the differences becomes negative, (R2) evaluate all values G_n on a set like $\tilde{\mathfrak{K}} := [0.9\tilde{n}, 1.1\tilde{n}] \cap \mathbb{N}$, and pick the largest value, (R3) use a Golden-Section-Search on $\tilde{\mathfrak{K}}$. Note, for $A = 20$ the optimal order quantity n^* equals 4, while for $A = 20,000$ this value equals 3866. Obviously, if n^* is "large" it is better to use procedure (R3). Procedure (R2) can be refined as follows: (1) in a first iteration choose $\tilde{\mathfrak{K}}$ to be small, e.g. $[0.99\tilde{n}, 1.01\tilde{n}] \cap \mathbb{N}$ and, if necessary, shift the set to the right or to the left a finite number of times. For "small" values of $\left(\frac{\varepsilon-1}{\varepsilon c}\right)^{\varepsilon} A$ it does not really matter which procedure will be applied. The linearity (in A) of the guess $\tilde{n} = \left(\frac{\varepsilon-1}{\varepsilon c}\right)^{\varepsilon} A$, see recommendations (i) and (ii), can be used to extrapolate the results of "small" problems to initialize search procedures when analyzing "large" problems.

V. CONCLUSIONS

This paper examined a nonstandard model of a passive price-setting newsvendor. The random demand, which depends on price, is not simply modeled in an additive fashion or a multiplicative one, see [10], but all moments of the demand distribution depend on the price variable. Also, in contrast to other studies, f.i. [11] and [12], the distribution is discrete and does not have compact support. We establish a closed form solution of the optimal value V^* and the optimal decisions n^* and p^* when the distribution is Poisson and the mean demand has constant elasticity. Approximations for these values are developed which are fairly good for reasonable demand elasticity parameters and large arrival rates. They have the appealing form of "optimal price $\approx$ markup $\times$ cost", and "order quantity $\approx$ expected demand". These approximations can be used to initialize numerical procedures when solving large newsvendor problems.

The paper then considered the situation of an active newsvendor, where the number of buyers is a controlled jump Markov process and the jump intensity has constant price elasticity. An active vendor becomes a passive one if he chooses the same price at every instance of time of the sales period. Since the price elasticity of demand is assumed to be constant a revenue maximizing active vendor is able to sell any number of items over any sales period (of positive length), see [5] and [8]. Thus, an active newsvendor will order a quantity for which marginal cost equates marginal revenue, and he will never have to deal with leftover items. In contrast, a passive vendor has to deal with situations when orders are too small or are too big.

However, so far we have not been able to derive a simple rule concerning optimal order quantities and optimal pricing decisions of passive newsvendors and active ones, cf. Table IV for a selection of different parameter values and optimal values.

TABLE IV

NUMERICAL VALUES FOR OPTIMAL STOCK LEVELS, OPTIMAL PRICES AND OPTIMAL PROFITS OF PASSIVE AND ACTIVE NEWSVENDOR MODELS.

		passive vendor			active vendor		
ε	A	n^*	p^*	$\bar{G}$	$\bar{n}$	$p^*(0,\bar{n})$	$\bar{H}$
1.5	20	4	3.32	5.5	5	3.09	6.4
2.0	20	5	1.96	3.2	5	2.22	4.0
3.0	20	5	1.47	1.7	6	1.55	2.3
1.5	1000	196	3.02	369.7	195	3.00	382.3
2.0	1000	250	2.00	237.4	251	2.00	248.0
3.0	1000	292	1.49	138.8	297	1.50	146.8

Numerical studies indicate that for different ranges of values of ε an active vendor will stock more items if customers are price sensitive, i.e. $\varepsilon > 2$. A sufficient condition, which implies that an active vendor will stock more units than a passive one, is $\beta_n - \beta_{n-1} \geq \mathfrak{z}_n - \mathfrak{z}_{n-1}$, $n \in \mathbb{N}$. Figure 6 is an example which illustrates this property on a finite set of integers if $\varepsilon = 3$. In the case of price sensitive customers, the active vendor can fully exploit his ability to adjust prices should the speed of sales slows down. The numerical studies, cf. Table IV, also support one's intuition that the values $\bar{G}$ and $\bar{H}$ are monotone decreasing in ε and that, at least for large values of n and *time homogeneous* market situations, the difference between $\bar{G}$ and $\bar{H}$ (percentage-wise) is relatively small. In particular, the advantage which an active newsvendor enjoys over a passive vendor is small if one concentrates on the return on sales.

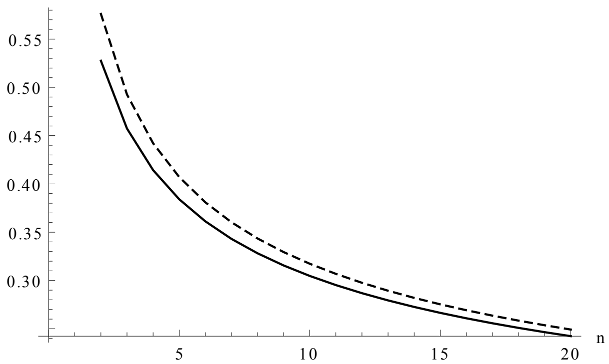

Fig. 6. Plot of the differences $(\beta_n - \beta_{n-1})$ (dashed line) and $(\mathfrak{z}_n - \mathfrak{z}_{n-1})$ (solid line). Parameters are $\varepsilon = 3$, $c = 1$ and $A = 20$.

Table II and additional numerical studies indicate that, for large values of n, the values β_n can be approximated by a simple convex combination of $\mathfrak{z}_n$ and $n^{(\varepsilon-1)/\varepsilon}$, namely

$$\beta_n \sim \mathfrak{z}_n/10 + 9/10 \cdot n^{(\varepsilon-1)/\varepsilon}.$$

Furthermore, the approximation $\tilde{p} = (c/\varepsilon - 1)c$ is the classical markup rule for a deterministic problem with demand function $\lambda = Ap^{-\varepsilon}$.

Finally, numerical procedures for passive newsvendor problems are considered. It was demonstrated by simple examples that for problems, where the optimal order quantity will be "large", a Golden-Section-Search method exploiting the unimodality of the profit sequence is fast and preferable over other methods. For "small" problems, it does not really matter which numerical method will be applied since all of them are equally fast. Estimates of the optimal order quantity are linear expressions in the shopping intensity. This scaling property can be exploited when the optimal decisions of problems with low shopping intensities are known, and problems with high shopping intensities, everything else being equal, have to be solved.

Among the many publications devoted to price-setting newsvendor problems, the analysis of Monahan et al.([9]) is most relevant for and related to our analysis. The authors of [9] consider a discrete time dynamic pricing problem with random demands, but allow any number (small or large) of time periods and quite general random demand processes; their random variables are not necessarily Poisson distributed. They approach this kind of pricing problem in a way which differs from the one which we adopt in Section II, and their approach leads to a reinterpretation of the dynamic pricing problem as a sequence of mildly coupled price-setting newsvendor problems. The mathematical techniques employed to analyze any of the subproblems or the problem at large are, however, totally different from the ones we employ to solve the one period [passive] newsvendor problem or the continuous time dynamic pricing problem. Besides the publications [5] and [8], which deal with either pure dynamic pricing or dynamic advertising and pricing problems (the first stage when analyzing active newsvendor problems) with iso-elastic demand, the paper by Gallego and van Ryzin [4] works with exponential rates, i.e. $\Lambda(t,p) = ae^{-bp}$. We intend to extend the present analysis to this case as well.

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