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Optimizing l_∞ Performance with Positivity Constraints

Mohammad Naghnaeian and Petros G. Voulgaris

I. Extended Abstract

There are many physical problems in which some variables are restricted be non-negative (or non-positive); examples can be found in biology, economics, and many other areas [1] [2] [3]. Motivated by such problems, the theory of positive systems has been the focus for many researchers. Notions such as stability, stabilizability, positive realization, and (distributed) control synthesis of such systems have been the subject of research [4]- [13].

We are interested in characterizing and optimizing the l_∞ gain of linear systems that contain positivity type of constraints. We first consider the case where the input is restricted to be in the positive cone of l_∞, denoted by l_∞^+, and seek to characterize the induced norm from l_∞^+ to l_∞. We call such an induced norm the *plus norm* and denote it by $\|.\|_+$. More precisely, for a LTV system T

$$\|T\|_+ = \sup_{\substack{u\neq 0 \\ u\in l_\infty^{n+}}} \frac{\|Tu\|_{l_\infty}}{\|u\|_{l_\infty}}.$$

We stress here, that no positivity constraint is imposed on the system itself. We obtain an exact characterization of this norm (the induced norm from l_∞^+ to l_∞) in terms of standard l_∞ induced norms of appropriately defined subsystems which is particularly easy to calculate in the case of LTI systems. To this end, we write T as the difference of positive operators T^+ and T^-, $T = T^+ - T^-$. We choose T^+ and T^- such that $[T^+]_{ij}[T^-]_{ij} = 0$, for any integers i and j, where by $[.]_{ij}$ we mean the ij^{th} entry in the infinite dimensional lower triangular matrix representation of the corresponding operator. Then, we have the following:

Lemma 1: For a linear time-varying operator T,

$$\|T\|_+ = \max\left\{\left\|T^+\right\|, \left\|T^-\right\|\right\},$$

where $\|.\|$ is the standard l_∞ induced norm.

Furthermore, in the case of linear time-invariant systems we have:

Theorem 2: Let $T : u \to y = (y_1, y_2, ..., y_n) \in l_\infty^n$ be LTI. Then,

$$\|T\|_+ = \max_i \frac{1}{2}\left[\|T_i\| + \left|\hat{T}_i(1)\mathbf{1}\right|\right],$$

where T_i is the LTI operator that maps u to y_i, $\hat{T}_i(1)$ is the λ-transform of T_i evaluated at 1, and $\mathbf{1}$ is the column vector of ones.

As an application (and motivation), we consider a filtering problem in which the signal to be estimated, s, is known to live in a positive cone, i.e. $s \in l_\infty^+$. In general, just designing a filter to minimize the standard l_∞ induced norm of the operator from signal to the estimation error will be conservative. Instead, we can use the apriori knowledge of positiveness of the signal by considering the same problem with l_∞^+ to l_∞ induced norm.

Furthermore, based on this development, it can be shown that:

1) Time-varying linear or nonlinear control or filtering does not improve the performance with respect to this norm for LTI systems.
2) The optimization is a linear programming problem.

We further generalize the results to the case of mixed input signals when there are inputs both in l_∞^+ and l_∞. As an example, consider the problem depicted in Figure 1, where for any non-negative k, $0 \leq s(k) \leq 1$ is the input to the plant P and

M. Naghnaeian is a PhD candidate with the Mechanical Science and Engineering Department, University of Illinois, Urbana, IL, USA naghnae2@illinois.edu

P. G. Voulgaris is with the Aerospace Engineering Department and the Coordinated Science Laboratory, University of Illinois, Urbana, IL, USA voulgari@illinois.edu

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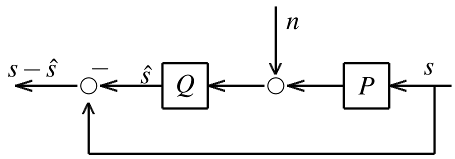

Fig. 1. Filtering problem

$-b \leq n(k) \leq b$, for some $b \geq 0$, is the measurement noise. The interest is to design a filter Q such that the difference between the input signal, s, and its estimate $\hat{s}$ is minimized in the l_∞ sense. This problem amounts to

$$\inf_{Q} \sup_{\substack{s\in\mathscr{B}\left(l_\infty^{1+},1\right)\\ n\in\mathscr{B}\left(l_\infty^{1},1\right)}} \|s-\hat{s}\|_{l_\infty} =$$

$$\inf_{Q} \sup_{\substack{s\in\mathscr{B}\left(l_\infty^{1+},1\right)\\ n\in\mathscr{B}\left(l_\infty^{1},1\right)}} \left\| \begin{bmatrix} I-QP & -bQ \end{bmatrix} \begin{pmatrix} s \\ n \end{pmatrix} \right\|_{l_\infty},$$

and it can be shown that

$$\inf_{Q} \sup_{\substack{s\in\mathscr{B}\left(l_\infty^{1+},1\right)\\ n\in\mathscr{B}\left(l_\infty^{1},1\right)}} \|s-\hat{s}\|_{l_\infty} =$$

$$\inf_{Q\in\mathscr{L}_{TI}} b\|Q\| + \frac{1}{2}\left[\|I-QP\| + \left|I-\hat{Q}(1)\hat{P}(1)\right|\right].$$

In the context of l_∞ optimization, we also consider the case when the output is forced to be in the positive l_∞ cone when the input is in this cone. This reflects as, so-called, an external positivity constraint on the system. As we point out, if such a constraint is imposed on the closed loop map, finding an optimal controller is a linear programming and hence a tractable problem [14]. If, on the other hand, the constraint known as internal positivity is sought, it can be shown that a dynamic controller offers no advantage over a static one. The results in [13] or [15] can be readily used to obtain an optimal (static) state feedback controller. We note that, designing an optimal output feedback controller (which is static) is a harder problem and in general leads to a bilinear program. We can also extend certain results of [13] to the case when part of the state is measurable (with measurement noise present).

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