

Geometric Structures for the Modelling, Analysis and Discretization of Infinite-Dimensional Port-Hamiltonian Systems

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Abstract— This mini-course is dealing with geometric structures that arise in the modelling of infinite-dimensional port-Hamiltonian systems. In four short lessons different geometric structures as well as their system theoretic consequences will be analyzed. This includes symplectic, polysymplectic as well as Stokes-Dirac structures, where the focus is put on the latter class, especially also with regard to diffusion systems and to higher order differential operators. Additionally, the structure preserving discretization by using simplicial Dirac structures will be discussed by applying tools from discrete exterior calculus. Finally, the connection to functional analytic concepts on Hilbert spaces will be presented.

I. INTRODUCTION

System analysis and control for systems described by partial differential equations (PDEs) is undoubtedly a challenging research area and a lot of progress has been achieved recently. Nevertheless, compared to the methods that are available in the case of ordinary differential equations (including the nonlinear case) the situation in the PDE case the situation is much more delicate and a lot of open question still need to be attacked. A promising direction in the infinite-dimensional scenario is to consider a sub-class of systems, such that the structure of the equations may be linked to the underlying physical principles. From the ODE case it is well-known that port-Hamiltonian systems possess this pleasing property (especially when the Hamiltonian corresponds to energy), see e.g. [20], therefore it is reasonable to follow this guide also in the PDE scenario, where (besides others) an additional challenge, the possibility of non-zero energy flow over the boundary, arises - a scenario that has no counterpart in the finite-dimensional case. From an interconnection and control viewpoint, such a treatment of boundary conditions is essential for the incorporation of energy exchange through the boundary, since in many applications the interconnection with the environment takes place precisely through the boundary.

Transferring this concept of the port-Hamiltonian picture from the finite-dimensional case to the infinite-dimensional case is not unique, but interesting is the observation, that geometric structures are the guide in most of the approaches, see e.g. [3], [12], [13], [21] and references therein. This severe problem how a port-Hamiltonian system in the PDE case should be formulated will be the starting point of the planned

mini-course, where we then subsequently focus on a specific approach, namely the approach based on Stokes-Dirac structures more closely, developed in [21], see also [11]. This method then also offers the possibility to analyze structure preserving discretization, [17] and additionally when Stokes-Dirac structures are considered on Hilbert spaces the link to functional analytic concepts becomes apparent, in particular connected to semi-group theory, [4], [8]. Regarding this latter point of view, we have submitted a second mini-course proposal (entitled: *Stability and Stabilization of Distributed Port-Hamiltonian Systems*), dealing exclusively with that aspect, whereas in this proposal it is planned to focus more on the geometric structures and its implications and only one of the four lessons shall provide the link to functional analytic concepts.

II. COURSE OUTLINE

This proposed mini-course shall consist of four lessons given by the following speakers:

- **Markus Schöberl** Post-Doc at the Institute of Automatic Control and Control Systems Technology, Johannes Kepler University, Linz, Austria
- **Bernhard Maschke** Professor in Automatic Control, University Claude Bernard Lyon 1, France
- **Marko Seslija** Post-Doc at the Department for Elektrotechnik, Katholieke University, Leuven, Belgium
- **Orest Iftime** Associate Professor in Applied Mathematics, University of Groningen, Netherlands

Each talk will be 25 minutes and the titles of the talks are:

- *Modelling of infinite-dimensional systems in a Hamiltonian framework*, by Markus Schöberl
- *Stokes-Dirac structures arising in distributed parameter systems*, by Bernhard Maschke
- *Structured discretization of port-Hamiltonian systems*, by Marko Seslija
- *Dirac Structures on Hilbert Spaces*, by Orest Iftime

The point of departure will be an introduction to modeling concepts in a port-Hamiltonian framework for infinite-dimensional systems. It is the aim to characterize different geometric concepts, such as symplectic structures, polysymplectic structures and Stokes-Dirac Structures, where a running example shall be used to highlight the different geometric concepts involved in these approaches. Additionally, the connection to variational calculus is stated, where it should be noted that of course the mathematical physics community

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offers Hamiltonian counterparts to Lagrangian field theory, but mainly by neglecting non-zero power flow over the boundary - to include this important effect was one of the motivations in the control community to propose different approaches, see e.g. [21] for the Stokes-Dirac approach, or [2], [14] for an alternative one based on jet-bundles. Additionally, system analytic concepts such as Casimir functionals used e.g. in the control by interconnection method for PDEs, see e.g. [9], [13] will be presented in the various scenarios. This introductory talk will be given by *Markus Schöberl*.

Subsequently, *Bernhard Maschke* will extensively discuss the Stokes-Dirac approach, where first the role of systems of conservation laws in the Port-Hamiltonian framework defined on Stokes-Dirac structure is illustrated (by using several physical examples), see [11]. Furthermore, dissipative systems such as diffusion systems are analyzed and it will be shown that can also be represented as dissipative port Hamiltonian systems defined on Stokes-Dirac structures. Then, the case of higher-order differential Hamiltonian operators is discussed in relation with fluid dynamical models such as the Korteweg de Vries equations, see e.g. [10]. As indicated above this second lesson is devoted in particular to the approach based on Stokes-Dirac structures, presenting extensions to the original work in [21], and this formulation will then also be heavily used in the final two lessons dealing with discretization and the relation to functional analytic concepts by introducing Stokes-Dirac structures on Hilbert spaces.

For numerical integration, simulation and control synthesis, it is of great interest to have finite approximations of distributed-parameter port-Hamiltonian systems that can be interconnected to one another or via the boundary coupled to other systems, be they finite- or infinite-dimensional. Most of the numerical algorithms for spatial discretization of distributed-parameter systems, (primarily finite difference and finite element methods), fail to capture the intrinsic system structures and properties. Furthermore, some important results, including Stokes theorem, fail to apply numerically. Given the overwhelming geometric nature of port-Hamiltonian systems, the difficulty to preserve some inherently topological and geometric structures of the continuous models gives a motivation to approach computations from a geometric standpoint. Thus, in the third lesson *Marko Seslija* will investigate *finite* approximations of distributed-parameter port-Hamiltonian systems. A framework for structure-preserving discretization of open distributed-parameter systems with generalized Hamiltonian dynamics will be provided, [16], [17]. Employing the formalism of discrete exterior calculus, [1], [5] simplicial Dirac structures will be introduced as discrete analogues of the Stokes-Dirac structure and demonstrated that they provide a natural framework for deriving finite-dimensional port-Hamiltonian systems that emulate the behaviors of their infinite-dimensional counterparts.

As indicated already in the introduction, the organizers of this mini-course also have prepared a second proposal concerning stability and stabilization of port-Hamiltonian

in the PDE scenario. Indeed, it is evident that whenever the stability proof of controlled port-Hamiltonian systems are in the scope techniques from functional analysis are vitally important - this gives an very stimulating interplay between the geometric nature of the PDEs (since they are modeled in a port-Hamiltonian framework) and the solutions of the systems which have to be analyzed in a different framework, namely using functional analysis, [8]. Thus the final lesson, given by *Orest Iftime*, will focus on Stokes-Dirac structure on Hilbert spaces, which could serve as an appetizer for the second mini-course as indicated already above that should deal exclusively with functional analytic concepts since stability issues are in the focus, [6], [7].

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