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Mini-course on polynomial optimization and control

by
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1 Topic

The mini-course focuses on polynomial optimization and control, with a focus on semidefinite relaxation techniques exploiting duality between moment problems and representations of polynomials nonnegative on semialgebraic sets.

In the early 2000s, several research groups realized that the linear matrix inequality (LMI) framework existing for analysis and control of linear dynamical systems [3] can be extended significantly to nonlinear control systems described by polynomial vector fields. Lyapunov analysis techniques can be extended readily as soon as one notices that positivity constraints on polynomials can be enforced by sum-of-squares (SOS) constraints on polynomials [22]. It turns out that SOS polynomials are semidefinite representable [20], i.e. they can be modelled by projecting a linear section of the semidefinite cone, or cone of nonnegative quadratic forms. In turn, optimization over the semidefinite cone can be efficiently carried out with interior-point methods [1]. Building on results of functional analysis and real geometric geometry allowing SOS representations of polynomials positive on semialgebraic sets [23], a hierarchy of LMI relaxations was proposed in [15] for polynomial optimization (minimization of a real-valued polynomial subject to finitely many polynomial inequalities and equations), with convergence guarantees. Of particular importance is the duality between the cone of nonnegative measures (resp. the cone of truncated moment sequences) and the cone of nonnegative functions (resp. the cone of nonnegative polynomials). The cone of truncated moment sequences is approximated from the exterior (relaxed) with appropriate linear sections of the semidefinite cone, whereas the cone of nonnegative polynomials is approximated from the interior (strengthened) with appropriate linear sections of the semidefinite cone.

Public-domain software packages were then released to illustrate the potential of these techniques to solve non-trivial optimization and control problems, see e.g. [7, 8]. Applications in control were surveyed in 2005 in the collective work [9]. The approach was later on extended to polynomial optimal control and infinite-dimensional optimization [16]. After more than a decade, these ideas have been consolidated in a comprehensive survey [19], research monograph [17], a control-oriented survey [4], a collective work reporting on the achievements of a related NSF funded project [2], and more recently, tutorial lecture notes [12].

The objective of this mini-course is to introduce these ideas in a unified, yet accessible fashion, with the hope that it can stimulate further research activities along these lines.

2 Outline

2.1 1st block (more theoretical)

2.1.1 1st talk (50 min) by Didier Henrion: Semidefinite programming relaxations of nonconvex problems of polynomial optimization and optimal control

This introductory lecture explains the basics of the hierarchy of LMI relaxations for polynomial optimization [15], focusing on the primal problem of moments and the dual problem of nonnegative polynomials. More recent extensions and developments are sketched.

2.1.2 2nd talk (50 min) by Minai Putinar: Representations of nonnegative functions on semialgebraic sets and related problems of moments

This lecture explains the core mathematical components instrumental to the proof of convergence of the hierarchy of LMI relaxations [23]. Here too the presentation emphasizes the duality relationships between the primal problem of deciding whether a real sequence are moments of a Borel measure, and the dual problem of deciding whether a polynomial is nonnegative on a basic closed semialgebraic set. Extensions to representations of semialgebraic functions are described [18].

2.2 2nd block (more applied)

2.2.1 3rd talk (50 min) by Mathieu Claeys: Numerical methods for designing relaxed control laws in nonlinear optimal control

After explaining the basic ideas [16] allowing to extend the hierarchy of LMI relaxations from static polynomial optimization to dynamic polynomial optimization (that is, infinite-dimensional calculus of variations [5], optimal mass transportation [24], and optimal control [6]), this lecture describes the software components [11] used to solve numerically, with convergence guarantees, challenging problems of optimal control, especially those with no solutions in classical Lebesgue spaces.

2.2.2 4th talk (50 min) by Milan Korda: Numerical methods for computing regions of attraction and maximal controlled invariant sets of nonlinear systems

Finally, this lecture reports on the most recent extensions of the approach of [16], based on results of [10], allowing the numerical computation of regions of attraction [14] and maximal controlled invariant sets for nonlinear systems [13].

3 Lecturers

3.1 Didier Henrion

D. Henrion is a CNRS researcher working at LAAS in Toulouse, France. He is also a Professor at the Faculty of Electrical Engineering of the Czech Technical University in Prague, Czech Republic. He is interested in polynomial optimization for systems control, focusing on the development of constructive tools for addressing mathematical problems arising from systems control theory. See `homepages.laas.fr/henrion`

3.2 Mihai Putinar

M. Putinar is a Professor at the Department of Mathematics of the University of California at Santa Barbara, USA, and at the Nanyang Technological University in Singapore. He is an expert in operator theory, functional analysis (problems of moments) and real algebraic geometry (representation of nonnegative polynomials). See `www.math.ucsb.edu/~mputinar`

3.3 Mathieu Claeys

M. Claeys is a Research Associate at the Department of Engineering of the University of Cambridge, UK. Before that, he defended a PhD thesis on occupation measures for polynomial optimal control at LAAS in Toulouse, France, under the supervision of Didier Henrion and Jean-Bernard Lasserre. See `homepages.laas.fr/mclaeys`

3.4 Milan Korda

M. Korda is a PhD student at Ecole Polytechique Fédérale de Lausanne, Switzerland. He has collaborated with C. Jones and D. Henrion on the topic of LMI relaxations for nonlinear systems control. See `people.epfl.ch/milan.korda`

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