Well-posed systems - linear and with nonlinear feedback

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NOTE: THIS IS NOT A PAPER, BUT A MINI-COURSE PLAN.

This minicourse is about well-posed systems, intended for newcomers to the field. Thus, no prior background on well-posed systems is assumed and we intend to guide the reader through the many concepts and available results, explaining their origin and significance as best as we can in the limited time. We assume that the reader has a basic understanding of functional analysis and operator semigroups, as well as finite-dimensional linear systems.

Informally speaking, a system is *well-posed* if on any time interval $[\tau, t]$, for any initial state x_0 in the state space and any input function u in a specified space of functions, it has a unique state trajectory x and a unique output function y, both defined on $[\tau, t]$. Moreover, y must belong to a specified space of functions, and both $x(t)$ and y must depend continuously on $x(\tau)$ and on u. This concept is general and can be made precise for many classes of non-linear and/or time-varying systems. However, most attention in the literature has been devoted to the simplest particular case, namely, linear and time-invariant (LTI) systems, because here we have strong tools to develop the theory. In the LTI context, if the state space is finite-dimensional, then well-posedness is not an issue and is usually not even mentioned. The theory focuses on systems with an infinite-dimensional state space, usually a Hilbert space. This is motivated by a variety of systems described by partial differential or delay equations, that can be shown to fit into this framework. Establishing well-posedness is usually not a goal in itself but opens the way for dealing with control and/or estimation problems by trying to mimick the rich finite-dimensional control theory using "operators in place of matrices".

Parts of this minicourse will stay close to the survey paper "Well-posed systems - the LTI case and beyond" which has been submitted in 2013 by Marius Tucsnak and George Weiss, while other parts will stay close to the book "Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces", by Birgit Jacob and Hans Zwart, Operator Theory, Advances and Applications vol. 223, Birkhäuser Verlag, Basel, 2012.

1 Well-posed LTI systems in Hilbert spaces

Speaker: George Weiss. Time: 25 minutes

In this part we review the linear theory. First we give a brief overview of the main facts known about well-posed linear time-invartiant systems in the Hilbert space context. We give the motivation and introduce the concept, after which we discuss the representation of such systems via a semigroup generator A, a control operator B, an observation operator C and a transfer function $\mathbf{G}$. We recall the admissibility concepts for unbounded B and C.

Next we introduce the larger class of LTI systems known as system nodes. This is a simple and very useful conncept when we model physical systems, or when we introduce special classes of systems, as there are almost no well-posedness assumptions involved, and well-posedness can be checked at a later stage. We introduce the concepts of classical and generalized solution of the system equations, and discuss their properties, in particular in the well-posed case.

We introduce regular linear systems, a subclass of the well-posed ones for which there is a well-defined feedthrough operator, that expresses the instantaneous effect of the input signal on the output signal. The feedthrough operator (if it exists) is the strong limit of the transfer function at $+\infty$. We recall different equivalent ways to express regularity, and a simpler way to to write the system equations.

Finally, we introduce impedance passive system nodes, scattering passive (hence well-posed) systems and scattering conservative systems (all in the LTI context). For instance, *scattering passive systems* are those that satisfy the power balance inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leqslant \|u(t)\|^2 - \|y(t)\|^2 .$$

along each classical solution, where u is the input function, x is the state trajectory and y is the output function. *Scattering energy preserving systems* are those for which we always have equality in the above formula. *Scattering conservative systems* are those scattering energy preserving systems whose dual is also scattering energy preserving. These properties can be checked via algebraic conditions that have to be satisfied by the system operators.

If time permits, we illustrate the theory with examples involving the heat and wave equations.

2 Well-posedness using energy methods

Speaker: Hans Zwart. Time: 25 minutes

We give some well-posedness results for system nodes that are derived from energy considerations. For example, a scattering passive system node is always well-posed. An impedance passive system node is well-posed if and only if its transfer function is bounded on a vertical line in the right half-plane.

For the port-Hamiltonian class of systems on a one-dimensional spatial domain, we explore the well-posedness in more detail. A linear port-Hamiltonian system on the one dimensional spatial domain (a,b) is described by the following PDE:

$$\frac{\partial x}{\partial t}(\zeta,t) = P_N \frac{\partial^N}{\partial \zeta^N}(\mathscr{H}x)(\zeta,t) + \cdots$$

$$+ P_1 \frac{\partial}{\partial \zeta}(\mathscr{H}x)(\zeta,t) + P_0(\mathscr{H}x)(\zeta,t),$$

with an initial condition $x(0,\zeta) = x_0(\zeta)$. Here ζ denotes the spatial coordinate, $P_k^* = (-1)^{k+1}P_k$ and $\mathscr{H}(\zeta) \geqslant mI$. The input and output are formulated via the boundary values, i.e., via $\frac{\partial^k}{\partial \zeta^k}(\mathscr{H}x)(a,t)$ and $\frac{\partial^k}{\partial \zeta^k}(\mathscr{H}x)(b,t)$ with $0 \leqslant k \leqslant N-1$.

Well-posedness means here that the inhomogeneous PDE possesses for every square integrable input a unique continuous state trajectory and the corresponding output is square integrable. If $N=1$, then there exist simple necessary and sufficient conditions characterizing well-posedness. For $N > 1$ the situation is much more unclear. In this presentation we concentrate on the important case $N = 2$ which includes the Schrödinger equation, and the Euler Bernoulli beam model.

3 Special classes of well-posed systems

Speaker: George Weiss. Time: 25 minutes

In the modeling of physical systems, we often encounter systems with a special structure. By analysing these special structures at an abstract level, we can get results about well-posedness, regularity, passivity, feedback stabilization, optimal control that can then be applied to any system in the class. In this part we describe three such special structures. As an illustration, we sketch here one of these:

Let H be a Hilbert space and assume that $A_0 : \mathscr{D}(A_0) \to H$ is positive and boundedly invertible operator. We introduce the scale of Hilbert spaces H_α, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geqslant 0$, $H_\alpha = \mathscr{D}(A_0^\alpha)$, with the norm $\|z\|_\alpha = \|A_0^\alpha z\|_H$. The space $H_{-\alpha}$ is defined as the dual of H_α with respect to the pivot space H. The operator A_0 can be extended (or restricted) to each H_α, such that

$$A_0 : H_\alpha \to H_{\alpha-1} \qquad \forall\, \alpha \in \mathbb{R}.$$

Let $C_0 \in \mathscr{L}(H_{\frac{1}{2}}, U)$, where U is another Hilbert space. We identify U with its dual and we denote $B_0 = C_0^*$, so that $B_0 : U \to H_{-\frac{1}{2}}$.

We consider the system described by

$$\ddot{z}(t) + A_0 z(t) + \frac{1}{2} B_0 \frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t) = B_0 u(t), \qquad (3.1)$$

$$y(t) = -\frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t) + u(t). \qquad (3.2)$$

The state $x(t)$ of this system and its state space X are defined by

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}, \qquad X = H_{\frac{1}{2}} \times H.$$

For classical solutions, we can rewrite the equations (3.1), (3.2) as a first order system as follows:

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= \overline{C}x(t) - u(t), \end{cases}$$

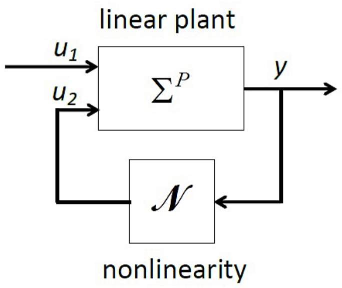

Figure 1. The nonlinear infinite-dimensional system $\Sigma^{\mathscr{N}}$ obtained from the linear system Σ^P by static output feedback through $\mathscr{N}$.

where

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -\frac{1}{2}B_0C_0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix},$$

$$\mathscr{D}(A) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \;\middle|\; A_0 z + \frac{1}{2}B_0C_0 w \in H \right\},$$

$$\overline{C} : H_{\frac{1}{2}} \times H_{\frac{1}{2}} \to U, \qquad \overline{C} = [0 \;\; C_0].$$

It is not difficult to check that A is m-dissipative. We denote by C the restriction of $\overline{C}$ to $\mathscr{D}(A)$ and for all $s \in \mathbb{C}_0$ we define

$$\mathbf{G}(s) = \overline{C}(sI - A)^{-1}B - I$$

$$= C_0 s \left(s^2 I + A_0 + \frac{s}{2}B_0C_0\right)^{-1} B_0 - I.$$

Proposition 3.1. *With the above notation, $(A, B, C, \mathbf{G})$ is a scattering conservative system node on (U, X, U).*

4 Feedback - linear and nonlinear

Speaker: Marius Tucsnak. Time: 25 minutes

First we recall the basic facts of the linear feedback theory developed for well-posed (and in particular, for regular) linear systems.

Then we go beyond the linear framework by exporing well-posedness results for well-posed linear systems with static nonlinear feedback. This part will contain several new results and applications. We consider well-posed linear systems with two components of the input and a static nonlinear output feedback going to the secons input, as shown in the figure.

Let U_1, U_2, X and Y be Hilbert spaces, and $U = U_1 \oplus U_2$. Let $\Sigma^P = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$ be a well-posed linear system with input space U, state space X and output space Y. In this minicourse description, for lack of space, we have to assume that the reader knows this concept. The operators Φ_τ and $\mathbb{F}_\tau$ can be decomposed according to the above decomposition of U: $\Phi_\tau = [\Phi_\tau^1 \; \Phi_\tau^2]$ and $\mathbb{F}_\tau = [\mathbb{F}_\tau^1 \; \mathbb{F}_\tau^2]$. The transfer function of Σ^P can be decomposed similarly: $\mathbf{G} = [\mathbf{G}_1 \; \mathbf{G}_2]$. We denote by $\omega_0(\mathbb{T})$ is the growth bound of $\mathbb{T}$.

Let $\mathscr{N} : Y \to U_2$ be a Lipschitz map with Lipschitz constant L. The feedback interconnection of Σ^P and $\mathscr{N}$, denoted by $\Sigma^{\mathscr{N}}$, is the dynamic system obtained by imposing that u_2, the second component of u, is obtained from y via $\mathscr{N}$:

$$u_2(t) = \mathscr{N} y(t) \qquad \forall\, t \in [0, \infty).$$

The system $\Sigma^{\mathscr{N}}$, shown as a block diagram in Figure 1, is said to be *well-posed* if for any input $u_1 \in L^2_{\text{loc}}([0,\infty); U_1)$ and any initial state $z_0 \in X$, there exist unique functions $z \in C([0,\infty); X)$ (the state trajectory) and $y \in L^2_{\text{loc}}([0,\infty); Y)$ (the output function) that satisfy

$$z(t) = \mathbb{T}_t z_0 + \Phi_t^1 u_1 + \Phi_t^2 \mathscr{N} y, \tag{4.1}$$
$$\mathbf{P}_t y = \Psi_t z_0 + \mathbb{F}_t^1 u_1 + \mathbb{F}_t^2 \mathscr{N} y, \tag{4.2}$$

for all $t \geqslant 0$, and moreover, on any bounded time interval $[0, \tau]$, $z(\tau)$ and $\mathbf{P}_\tau y$ depend continuously on z_0 and on $\mathbf{P}_\tau u_1$. The continuous dependence is meant with respect to the usual norm for states, and with respect to the L^2 norm for the input and output functions.

Theorem 4.1. *With the notation of this section, if*

$$\inf_{\omega > \omega_0(\mathbb{T})} \|\mathbb{F}^2_\infty\|_\omega L < 1,$$

where L is the Lipschitz constant for the nonlinearity, then $\Sigma^{\mathscr{N}}$ is well-posed.

A more sophisticated result involves a feedback that is a bilinear function of the state and the output: $u_2 = \mathscr{N}(x, y)$. All these results are of small gain type, and they guarantee the well-posedness of the closed-loop system for certain Lipschitz constants of the nonlinearity, or the local well-posedness for bilinear feedbacks satisfying a certain estimate. Using this nonlinear feedback theory we prove the global well-posedness of a system described by Burgers equation and the local well-posedness of the Navier-Stokes equations. The results are technical and would need a too much space to state them here.

This minicourse *will not* cover the following topics: exact controllability and exact observability, stability and stabilization, optimal control and optimal estimation. Indeed, these topics are not directly related to well-posedness (even though they use results about it). To our regret, because of the time constraints, we also have to leave out the Lax-Phillips semigroup associated to a well-posed system (the connection between well-posed systems theory and scattering theory), as well as time-varying well-posed linear systems.